Chapter 10

Dynamics of Rotational Motion
Torque

- Definition:
  - A force can be defined as something that causes linear motion to change.
  - A torque can be similarly defined as something that causes a change in rotational motion.
• The torque applied to a rigid body depends on the magnitude of the applied force, the distance away from the point of rotation (lever arm), and the direction of the applied force.

• We can express the torque as a vector cross product.
Example

- The Achilles tendon of a person is exerting a force of magnitude 720 N on the heel at a point that is located $3.6 \times 10^{-2}$ m away from the point of rotation.
- Determine the torque about the ankle (point of rotation).
- Assume the force is perpendicular to the radial arm.
Picture of foot and Achilles tendon

- Force $F$
- Angle $55^\circ$
- Distance $3.6 \times 10^{-2} \text{ m}$
Solution

• The magnitude of the torque is:
Torque and Angular Acceleration

• Consider a point on a rigid body.
• If a force is applied we can determine the tangential acceleration by Newton’s second law.
• We can express this in terms of the angular acceleration by the following relation.

• The force on the particle can now be written as:
• If we apply the definition of torque to our previous relation we get the following:

• We can use the vector triple cross product identity to rewrite this expression.
Vector Triple Cross Product

\[ \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B} (\vec{A} \cdot \vec{C}) - \vec{C} (\vec{A} \cdot \vec{B}) \]
• If we apply this identity to our product we get the following:

• However, $r$ and $\alpha$ are perpendicular; therefore,
• The torque now becomes:

• We can rewrite this in terms of the moment of inertia.
Rigid-Body Rotations About a Moving Axis

• Consider the motion of a wheel on an automobile as it moves along the road.

• Since the wheel is in motion it possesses kinetic energy of motion.

• However, since the wheel is both translating along the road and rotating, the kinetic energy of the wheel is shared between these two motions.
• The kinetic energy of a rotating body about a moving axis is given by:
Example

- Determine the kinetic energy of a solid cylinder of radius $R$ and mass $M$ if it rolls without slipping.
Solution

• If an object rolls without slipping the speed of the center of mass is related to the angular speed by the following:
• The kinetic energy is given by:

• The moment of inertia for a solid cylinder is:
• The kinetic energy then becomes:
Work and Power for Rotations

• The work done rotating a rigid object through an infinitesimal angle about some axis is given by the following:
• The total work done rotating the object from some initial angle to some final angle is:
• If we examine the integrand we see that it can be rewritten as:
• We integrate to get the work done.
Power

• The power of a rotating body can be obtained by differentiating the work.
• If we differentiate we get:
Angular Momentum

• We define the linear moment of a system as:

• We saw that Newton’s second law can be written in terms of the linear momentum.
• We can define momentum for rotations as well.
• We define the angular momentum of a system as:

\[ \text{The value of the angular momentum depends upon the choice of origin.} \]
• The units for angular momentum are:

\[ \text{kg} \cdot \text{m}^2 / \text{s} \]
Newton’s Second Law

• Suppose an object is subjected to a net torque.
• How does this affect the angular momentum?
Suppose the angular momentum changes with time.

We can write the following:
• Now suppose the mass is constant.
• When we differentiate we get:
• The first term is equal to zero due to the properties of the cross product.
• Therefore, the change in angular momentum is:
Thus, we have Newton’s second law for rotations.
• The angular momentum around a symmetry axis can be expressed in terms of the angular velocity.
• If we take the time derivative of the angular momentum, once again we arrive at Newton’s second law.

\[ I \ddot{\alpha} = \tau \]
Example

• A woman with mass 50-kg is standing on the rim of a large disk (a carousel) that is rotating at 0.50 rev/s about an axis through its center.
• The disk has mass 110-kg and a radius 4.0m.
• Calculate the magnitude of the total angular momentum of the woman plus disk system.
• Treat the woman as a point particle.
Solution

• The angular momentum of the system is:

• The magnitude of the angular momentum of the system is:
• The magnitude of the linear momentum of the woman is:

• The magnitude of the angular momentum of the woman is:
• The magnitude of the angular momentum of the disk is:
• The magnitude of the total angular momentum of the system is:
The Conservation of Angular Momentum

• Consider a system with no net torque acting upon it.

• The equation above implies that if the net torque on a system is zero, then the angular momentum of the system will be constant.