Additional Problems 1

February 12, 2017

(1)
Up to an overall phase, an arbitrary $2 \times 2$ unitary matrix can be written as

$$U(\alpha, \beta, \gamma) = \begin{pmatrix}
    e^{-i(\alpha+\gamma)/2} \cos \beta/2 & -e^{-i(\alpha-\gamma)/2} \sin \beta/2 \\
    e^{-i(-\alpha+\gamma)/2} \sin \beta/2 & e^{i(\alpha+\gamma)/2} \cos \beta/2
\end{pmatrix} \quad \alpha, \beta, \gamma \in \mathbb{R}.
$$

What are the eigenvalues of and eigenvectors of $U(\alpha, \beta, \gamma)$? Consider all possible values for $(\alpha, \beta, \gamma)$.

**SOLUTION:**

To compute the eigenvalues, we use the general formula proven in (2a) below. That is,

$$\lambda_{\pm} = \frac{1}{2} \left( 2 \cos[(\alpha + \gamma)/2] \cos(\beta/2) \pm \sqrt{\cos^2[(\alpha + \gamma)/2] \cos^2(\beta/2) - 4} \right) \quad \alpha, \beta, \gamma \in \mathbb{R}.
$$

(1)

Notice that $|\lambda_{\pm}| = 1$. To find the eigenvectors, we compute

$$\begin{pmatrix}
    e^{-i(\alpha+\gamma)/2} \cos \beta/2 & -e^{-i(\alpha-\gamma)/2} \sin \beta/2 \\
    e^{-i(-\alpha+\gamma)/2} \sin \beta/2 & e^{i(\alpha+\gamma)/2} \cos \beta/2
\end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \lambda_{\pm} \begin{pmatrix} 1 \\ x \end{pmatrix}.
$$

The two values of $x$ for the two eigenvectors are

$$\begin{align*}
    x_{\pm} &= \frac{-\lambda_{\pm} - e^{-i\gamma/2} \cos(\beta/2)}{e^{-i(\alpha-\gamma)/2} \sin(\beta/2)} \\
    &= \frac{-e^{i\alpha/2} \lambda_{\pm} - e^{-i\gamma/2} \cos(\beta/2)}{e^{i\gamma/2} \sin(\beta/2)}.
\end{align*}
$$

(3)

(2)

An arbitrary $2 \times 2$ matrix $A$ can be written as

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, d \in \mathbb{C}.
$$

(a) Show that the two eigenvalues of $A$ are $\lambda_{\pm}$, where

$$\lambda_{\pm} = \frac{1}{2} \left( \text{Tr}(A) \pm \sqrt{\text{Tr}(A)^2 - 4 \det(A)} \right).
$$

(4)
The eigenvalues are solutions to the equation
\[
0 = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc = \lambda^2 - \lambda(a + d) + (ad - bc) = \lambda^2 - \lambda \text{Tr}(A) + \det(A).
\]

The quadratic formula for \( \lambda \) gives us the desired result.

(b) Show that \( \text{Tr}(A) = \lambda_+ + \lambda_- \) and \( \det(A) = \lambda_+ \lambda_- \).

SOLUTION:
These identities are straightforward to show from the previous equation.

(c) Prove that \( A\sigma_y A^T = \det(A)\sigma_y \),
where \( \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \).

SOLUTION:
We compute
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = i \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -b & -d \\ a & c \end{pmatrix} = i \begin{pmatrix} 0 & -ad + bc \\ -bc + ad & 0 \end{pmatrix} = \det(A)\sigma_y.
\]

(a) Prove that \( P = A^\dagger A \) is a positive operator for any operator \( A \).

SOLUTION:
To prove that \( P \) is positive, we need to show that \( \langle \psi | P | \psi \rangle \geq 0 \) for all \( |\psi\rangle \). Let \( |\psi'\rangle = P |\psi\rangle \). Then
\[
\langle \psi | P | \psi \rangle = \langle \psi | P^\dagger P | \psi \rangle = \langle \psi' | \psi' \rangle \geq 0,
\]
where we’ve used the fact that \( P = P^2 = P^\dagger P \).

(b) Conversely, show that every positive operator \( P \) can be decomposed as \( P = A^\dagger A \) for some operator \( A \). Is this operator \( A \) unique? [[Hint: Use the spectral decomposition of \( P \).]]

SOLUTION:
The spectral decomposition allows us to write \( P = \sum_i \lambda_i |b_i\rangle\langle b_i| \) for orthonormal vectors \( |b_i\rangle \) and nonnegative eigenvalues \( \lambda_i \). Then define the operator \( A = \sum_i \sqrt{\lambda_i} |b_i\rangle\langle b_i| \) so that \( A^\dagger A = P \). This operator is not unique since \( P = B^\dagger B \) for \( B = UA \) for any unitary \( U \).
In class we defined a positive operator to be any hermitian operator $P$ satisfying $\langle \psi | P | \psi \rangle \geq 0$ for all $\mathcal{H}$. Prove that $P$ is positive if and only if it is hermitian and has nonnegative eigenvalues.

**SOLUTION:**

$(\Rightarrow)$ Suppose that $P$ is positive. Let $|e\rangle$ be an eigenvector of $P$ with eigenvalue $\lambda$. Then $0 \leq \langle e | P | e \rangle = \lambda \langle e | e \rangle = \lambda$, where the first equality follows from the positivity assumption of $P$. $(\Leftarrow)$ Suppose that $P$ is hermitian with nonnegative eigenvalues. Then $P$ has a spectral decomposition $P = \sum_i \lambda_i |b_i\rangle \langle b_i|$ with the $\lambda_i \geq 0$. Then for any $|\psi\rangle$, we have

\[ \langle \psi | P | \psi \rangle = \sum_i \lambda_i \langle \psi | b_i \rangle \langle b_i | \psi \rangle = \sum_i \lambda_i |\langle \psi | b_i \rangle|^2 \geq 0. \]

(5) Let $\{P_i\}_{i=1}^d$ be a complete set of orthogonal projectors. That is $P_i = P_i^\dagger$, $P_i^2 = P_i$, and $\sum_{i=1}^d P_i = I$.

(a) Prove that $P_i P_j = 0$ for all $i \neq j$.

**SOLUTION:**

For any $i \neq j$ and any $|\psi\rangle \in \mathcal{H}$, consider the vector $|\psi_{ij}\rangle = P_i P_j |\psi\rangle$. Now keep $j$ fixed, and consider the sum

\[ \sum_{i \neq j} \langle \psi_{ij} | \psi_{ij} \rangle = \sum_{i \neq j} \langle \psi | (P_i P_j)^\dagger P_i P_j |\psi\rangle = \sum_{i \neq j} \langle \psi | P_j P_i P_i P_j |\psi\rangle \]

\[ = \sum_{i \neq j} \langle \psi | P_j P_i P_i P_j |\psi\rangle = \langle \psi | P_j (I - P_j) P_j |\psi\rangle = 0. \] \hspace{1cm} (6)

This implies that $0 = \langle \psi_{ij} | \psi_{ij} \rangle$ for all $i \neq j$. Thus $P_i P_j |\psi\rangle = 0$, and as this holds for all $|\psi\rangle$ we must have that $P_i P_j = 0$.

(b) Prove that any vector $|\psi\rangle$ can be decomposed as $|\psi\rangle = \sum_{i=1}^d \alpha_i |\psi_i\rangle$, where $\alpha_i = \sqrt{p(i)}$, $\sqrt{p(i)} = \langle \psi | P_i |\psi\rangle$, and $|\psi_i\rangle = \frac{P_i |\psi\rangle}{\sqrt{p(i)}}$.

**SOLUTION:**

Since $\sum_i P_i = I$, we have

\[ |\psi\rangle = \sum_i P_i |\psi\rangle = \sum_i \alpha_i \frac{P_i |\psi\rangle}{\alpha_i} = \sum_i \alpha_i |\psi_i\rangle. \] \hspace{1cm} (7)

This completes the proof.

(6) Compute the matrix for $(\sigma_x \otimes \sigma_z + \sigma_z \otimes \sigma_x)^2$, where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. 

\[ \text{Compute the matrix for } (\sigma_x \otimes \sigma_z + \sigma_z \otimes \sigma_x)^2, \text{ where } \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
SOLUTION:

We compute

\[(\sigma_x \otimes \sigma_z + \sigma_z \otimes \sigma_x)^2 = (\sigma_x \otimes \sigma_x)^2 + (\sigma_x \otimes \sigma_z)(\sigma_z \otimes \sigma_x) + (\sigma_z \otimes \sigma_x)(\sigma_x \otimes \sigma_z) + (\sigma_z \otimes \sigma_x)^2\]

\[= \sigma_x^2 \otimes \sigma_z^2 + \sigma_x \sigma_z \otimes \sigma_z \sigma_x + \sigma_z \sigma_x \otimes \sigma_x \sigma_z + \sigma_z^2 \otimes \sigma_x^2\]

\[= I \otimes I + \sigma_z \sigma_x \otimes \sigma_x \sigma_z + \sigma_z \sigma_x \otimes \sigma_x \sigma_z + I \otimes I\]

\[= 2 (I \otimes I + \sigma_z \sigma_x \otimes \sigma_x \sigma_z)\]

\[= 2 \left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) - \left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{array}\right) \otimes \left(\begin{array}{cccc}
0 & 1 \\
-1 & 0
\end{array}\right)\]

\[= 2 \left(\begin{array}{cccc}
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 1
\end{array}\right).\]