Additional Problems 1

January 30, 2017

(1) An arbitrary \(2 \times 2\) unitary matrix can be written as
\[
U(\alpha, \beta, \gamma) = \begin{pmatrix}
e^{-i(\alpha+\gamma)/2} \cos \beta/2 & -e^{-i(\alpha-\gamma)/2} \sin \beta/2 \\
e^{-i(-\alpha+\gamma)/2} \sin \beta/2 & e^{i(\alpha+\gamma)/2} \cos \beta/2
\end{pmatrix}
\]
\(\alpha, \beta, \gamma \in \mathbb{R}\).

What are the eigenvalues of and eigenvectors of \(U(\alpha, \beta, \gamma)\)? Consider all possible values for \((\alpha, \beta, \gamma)\).

(2) An arbitrary \(2 \times 2\) matrix \(A\) can be written as
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]
\(a, b, c, d \in \mathbb{C}\).

(a) Show that the two eigenvalues of \(A\) are \(\lambda_{\pm}\), where
\[
\lambda_{\pm} = \frac{1}{2} \left( \text{Tr}(A) \pm \sqrt{\text{Tr}(A)^2 - 4|\det(A)|} \right).
\]

(b) Show that \(\text{Tr}(A) = \lambda_{+} + \lambda_{-}\) and \(\det(A) = \lambda_{+}\lambda_{-}\).

(c) Prove that
\[
A\sigma_y A^T = \det(A)\sigma_y,
\]
where \(\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\).

(3)

(a) Prove that \(P = A^\dagger A\) is a positive operator for any operator \(A\).

(b) Conversely, show that every positive operator \(P\) can be decomposed as \(P = A^\dagger A\) for some operator \(A\). Is this operator \(A\) unique? [[Hint: Use the spectral decomposition of \(P\).]]

(4)
In class we defined a positive operator to be any hermitian operator \(P\) satisfying \(\langle \psi | P | \psi \rangle \geq 0\) for all \(\mathcal{H}\). Prove that \(P\) is positive if an only if it is hermitian and has nonnegative eigenvalues.

(5)
Let \(\{P_i\}_{i=1}^{d}\) be a complete set of orthogonal projectors. That is \(P_i = P_i^\dagger, P_i^2 = P_i\), and \(\sum_{i=1}^{d} P_i = I\).

(a) Prove that \(P_i P_j = 0\) for all \(i \neq j\).

(b) Prove that any vector \(|\psi\rangle\) can be decomposed as \(|\psi\rangle = \sum_{i=1}^{d} \alpha_i |\psi_i\rangle\), where \(\alpha_i = \sqrt{p(i)}\), \(\sqrt{p(i)} = \langle \psi | P_i | \psi \rangle\), and \(|\psi_i\rangle = \frac{P_i |\psi\rangle}{\sqrt{p(i)}}\).

(6)
Compute the matrix for \((\sigma_x \otimes \sigma_z + \sigma_z \otimes \sigma_x)^2\), where \(\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) and \(\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\).