Entanglement Distillation of a Pure State

Summary of Last Lecture

For any bipartite pure state $|\Psi\rangle$ with entanglement $E(\Psi)$, there exists a multi-outcome LOCC protocol (Call this Protocol 1) that transforms

$$|\Psi\rangle \otimes m = \sum_{k=0}^{m} \sqrt{q_k} |\Psi_k\rangle \xrightarrow{\text{LOCC}} \{q_k, |\Psi_k\rangle\}$$

with $q_k = \binom{m}{k} p^k (1 - p)^{m-k}$ such that

1. Each $|\Psi_k\rangle$ is a maximally entangled state of dimension $\binom{m}{k}$,
2. The average post-measurement entanglement per copy of $|\Psi\rangle$ is

$$\frac{1}{m} \langle E \rangle = \frac{1}{m} \sum_{k=0}^{m} q_k E(\Psi_k)$$

and satisfies

$$h(p) - \frac{\log(m + 1)}{m} \leq \frac{1}{m} \langle E \rangle \leq h(p).$$
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Our goal now is to use this analysis to build a full protocol that completes the desired transformation $|\psi\rangle^\otimes m \xrightarrow{\text{LOCC}} \approx |\Phi^+\rangle^\otimes n$ with $\frac{n}{m} \rightarrow E(\psi)$.

Let $\epsilon > 0$ be some arbitrarily small number.

Our strategy will be to repeat Protocol 1 $t$ times, each time performed on a block of $m_0$ copies of $|\psi\rangle$, where $m_0$ is some integer such that

$$\left| \frac{1}{m_0} \langle E \rangle - h(p) \right| \leq \epsilon/2,$$

and $\langle E \rangle$ is the average entanglement generated from Protocol 1 performed on $|\psi\rangle^\otimes m_0$. 
Law of Large Numbers

Let \( S = \{x_1, x_2, \ldots, x_r\} \) be some finite set of real numbers. Let \( X \) be a random variable that takes on value \( x_k \in S \) with probability \( p_k \).

The expected (or average) value of \( X \) is given by \( \langle X \rangle = \sum_{k=1}^{r} p_k x_k \).

Suppose now that we have \( t \) independent instances of \( X \). In other words, we take \( t \) independent and identical samples from \( S \), each sample drawing \( x_k \) with probability \( p_k \).

Let \( X_1, X_2, \ldots, X_t \) be the outcome of these samples.

Law of Large Numbers: For every \( \epsilon > 0 \),

\[
\lim_{t \to \infty} Pr \left\{ \left| \frac{1}{t} \sum_{k=1}^{t} X_k - \langle X \rangle \right| > \epsilon \right\} = 0.
\]
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Apply the Law of Large Numbers to the \( t \) identical and independent iterations of Protocol 1.

Let \( E_i \) be the post-measurement entanglement performing Protocol 1 on the \( i^{th} \) block.
In other words, \( E_i := E(\Psi_{k_i}) = \log \left( \frac{m_0}{k_i} \right) \) when projector \( P_{k_i} \) is performed on the \( i^{th} \) block, thereby collapsing \( |\Psi\rangle \otimes m_0 \) into \( |\Psi_{k_i}\rangle \) in that block.

We know that the expected (or average) entanglement for each block is the same, and it is given by \( \langle E \rangle = \sum_k q_k E(\Psi_k) \).

The Law of Large Numbers:

\[
\exists T \text{ such that } t > T \implies \Pr \left\{ \left| \frac{1}{t} \sum_{i=1}^{t} \frac{E_i}{m_0} - \frac{\langle E \rangle}{m_0} \right| > \frac{\epsilon}{4} \right\} < \epsilon.
\]
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The next step is to choose $t$ large enough so that with high probability, the total entanglement $\sum_{i=1}^{t} E_i$ is equivalent to entanglement of the form $|\psi^+\rangle \otimes l$ with $\frac{1}{t} \sum_{i=1}^{t} \frac{E_i}{m_0}$ still being close to $\frac{\langle E \rangle}{m_0} \approx h(p)$.

**Claim:** For any $T, \epsilon > 0$, there exists an integer $t_0 > T$ and integer $l_0$ such that

$$\epsilon > \frac{\langle E \rangle}{m_0} - \frac{2^{l_0}}{t_0 m_0} > \frac{\epsilon}{4}.$$

**Proof:**
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Then by the Law of Large Numbers we know that with probability $\geq 1 - \epsilon$, when performing Protocol 1 on $t_0$ blocks of $|\psi\rangle \otimes m_0$ a post-measurement state

$$|\psi_{\vec{k}}\rangle := |\psi_{k_1}\rangle \otimes |\psi_{k_2}\rangle \otimes \cdots \otimes |\psi_{k_{t_0}}\rangle$$

is obtained with

$$\left| \frac{1}{t_0} \sum_{i=1}^{t_0} \frac{E_i}{m_0} - \langle E \rangle \right| > \epsilon/4.$$ 

Hence with probability $\geq 1 - \epsilon$,

$$\sum_{i=1}^{t_0} E_i \geq 2^{l_0}.$$
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Notice that $\sum_{i=1}^{t_0} E_i$ is the entanglement of

$$|\Psi_{\vec{k}}\rangle := |\Psi_{k_1}\rangle \otimes |\Psi_{k_2}\rangle \otimes \cdots \otimes |\Psi_{k_{t_0}}\rangle$$

by additivity of von Neumann entropy. The full state $|\Psi_{\vec{k}}\rangle$ is maximally entangled with dimension $\prod_{i=1}^{t_0} (m_0)_{k_i}$, and so

$$E(\Psi_{\vec{k}}) = \log \prod_{i=1}^{t_0} \binom{m_0}{k_i} = \sum_{i=1}^{t_0} \log \binom{m_0}{k_i} = \sum_{i=1}^{t_0} E_i.$$

Similarly $2^{l_0}$ is the entanglement of $|\Phi^+\rangle \otimes l_0$.

$|\Psi_{\vec{k}}\rangle$ is LU equivalent to $|\Phi^+\rangle \sum_{i=1}^{t_0} E_i$ and $|\Phi^+\rangle \otimes l_0$ is LU equivalent to $|\Phi^+_{2^{l_0}}\rangle$. 

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**Proposition:** If \( d \geq d' \), then it is always possible to transform \( |\Phi^+_d\rangle \) into \( |\Phi^+_{d'}\rangle \) with probability 1 using LOCC.

**Proof:** Let \( S \) be the collection of all subsets of \( \{1, \cdots, d\} \) that have \( d' \) distinct elements. In total, \( S \) contains \( \binom{d}{d'} \) subsets \( S_1, S_2, \cdots, S_{\binom{d}{d'}} \).

Moreover, each \( i \in \{1, \cdots, d\} \) will appear in exactly \( \binom{d-1}{d'-1} \) subsets in \( S \).

Therefore we can define the (non-orthogonal) projectors \( P_1, P_2, \cdots, P_{\binom{d}{d'}} \),

\[
P_k = \sum_{i \in S_k} |i\rangle\langle i|, \quad \Rightarrow \quad \sum_{k=1}^{\binom{d}{d'}} P_k = \left( \begin{array}{c} d - 1 \\ d' - 1 \end{array} \right) \mathbb{I}.
\]
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So the set of operators \( \{ M_k \}_{k=1}^{d,d'} \) with \( M_k = \sqrt{\frac{d-1}{d'-1}} \) is a valid generalized measurement that will transform \( |\Phi_d^+\rangle \rightarrow |\Phi_{d'}^+\rangle \) up to a local unitary transformation.

Applying this to the state \( |\Psi_k^\rightarrow\rangle \) and \( |\Phi^+\rangle \otimes l_0 \), we see that \( |\Psi_k^\rightarrow\rangle \) can be transformed into \( |\Phi^+\rangle \otimes l_0 \) using LOCC.

The entanglement of \( |\Phi^+\rangle \otimes l_0 \) satisfies \( \left| \frac{\langle E \rangle}{m_0} - \frac{2l_0}{t_0 m_0} \right| < \frac{\epsilon}{2} \), and by our original assumption we have \( \left| \frac{\langle E \rangle}{m_0} - h(p) \right| \leq \frac{\epsilon}{2} \).

Therefore Triangle Inequality implies \( \left| \frac{2l_0}{t_0 m_0} - h(p) \right| < \epsilon \).
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Here is the total protocol:

**Step 1:** Alice and Bob start with $|\psi\rangle^{t_0m_0}$. They perform Protocol 1 on blocks of size $m_0$.

With probability $1 - \epsilon$ they generate a state $|\psi_{\vec{k}}\rangle$ with

$$\left| \frac{1}{t_0} \sum_{i=1}^{t_0} \frac{E_i}{m_0} - \frac{\langle E \rangle}{m_0} \right| > \epsilon/4.$$

If they do not obtain such a state, they abort the protocol and prepare the product state $|\text{Fail}\rangle|\text{Fail}\rangle$, which is any state orthogonal to $|\Phi^+\rangle \otimes l_0$. 
Step 2: If they did not fail at the end of the previous step, then they convert $|\Psi_k\rangle$ into $|\Phi^+\rangle^{\otimes l_0}$ with $\left| \frac{2l_0}{t_0m_0} - h(p) \right| < \epsilon$.

In total they have implemented the transformation

$$|\Psi\rangle^{\otimes t_p m_0} \rightarrow \rho_{t_0 m_0} = (1 - \epsilon')|\Phi^+\rangle\langle\Phi^+|^{\otimes l_0} + \epsilon'|\text{Fail, Fail}\rangle\langle\text{Fail, Fail}|$$

with $\epsilon' \geq \epsilon$.

The fidelity of $\rho_{t_0 m_0}$ satisfies $F(\rho_{t_0 m_0}, |\Phi^+\rangle^{\otimes l_0}) \geq 1 - \epsilon$ and the entanglement per copy satisfies $\left| \frac{2l_0}{t_0m_0} - h(p) \right| < \epsilon$.

Since $\epsilon$ is arbitrary, this shows that $E_D(|\Psi\rangle\langle\Psi|) = E(\Psi)$. 