A Physical Meaning of Convex Roof-Extended Monotones

Before further exploring the entanglement entropy, let us try to provide a physical meaning of convex roof-extended monotones.

For a pure state entanglement monotone $\mu$ we have shown that the function

$$\hat{\mu}(\rho) = \min_{\{\varphi_i, p_i\}} \sum_i p_i \mu(|\varphi_i\rangle)$$

is a convex mixed-state entanglement monotone.

The minimization is taken over all pure state ensembles that generate $\rho$.

Two ensembles $\mathcal{E}_1 = \{|\psi_i\rangle, p_i\}_{i=1}^{r}$ and $\mathcal{E}_2 = \{|\psi'_i\rangle, p'_i\}_{i=1}^{r'}$ generating the same density matrix means that $\rho = \sum_{i=1}^{r} p_i |\psi_i\rangle \langle \psi_i| = \sum_{i=1}^{r'} p'_i |\psi'_i\rangle \langle \psi'_i|$. 
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Let \( \rho = \sum_{j=1}^{r} \lambda_i |e_i\rangle \langle e_i| \) be an eigenvalue decomposition of \( \rho \).

Recall that an ensemble \( \{|\varphi_i\rangle, p_i\}_{i=1}^{t} \) generates \( \rho \) iff there exists a \( t \times t \) unitary matrix \( U \) (with elements \( u_{ij} \)) such that

\[
\sqrt{p_i} |\varphi_i\rangle = \sum_{i=1}^{t} u_{ij} \sqrt{\lambda_j} |e_j\rangle,
\]

where \( |e_j\rangle = 0 \) for \( j > r \).

Each ensemble \( \mathcal{E} = \{|\varphi_i\rangle, p_i\}_{i=1}^{t} \) has an average value of \( \mu \) given by

\[
\langle \mu \rangle_{\mathcal{E}} = \sum_i p_i \mu(|\varphi_i\rangle).
\]

The value \( \hat{\mu}(\rho) \) gives the smallest average value of \( \mu \) among all ensembles generating \( \rho \).
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For every ensemble $\mathcal{E} = \{ |\varphi_i\rangle, p_i \}_{i=1}^t$ generating $\rho$, we can form the purification

$$|\omega_\mathcal{E}\rangle^{ABC} = \sum_{i=1}^t \sqrt{p_i} |\varphi_i\rangle^{AB} |i\rangle^C.$$ 

We have previously shown that any two purifications of the same mixed state must be related by a unitary transformation on the purifying system: If $tr_C (|\omega_{\mathcal{E}_1}\rangle\langle\omega_{\mathcal{E}_1}|^{ABC}) = tr_C (|\omega_{\mathcal{E}_1}\rangle\langle\omega_{\mathcal{E}_2}|^{ABC})$, then there exists some unitary matrix $U$ acting on system $C$ such that

$$|\omega_{\mathcal{E}_1}\rangle^{ABC} = \mathbb{I}^{AB} \otimes U^C |\omega_{\mathcal{E}_2}\rangle^{ABC}.$$
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If $\mathcal{E}_1 = \{|\psi_i\rangle, p_i\}^r_{i=1}$ and $\mathcal{E}_2 = \{|\psi'_i\rangle, p'_i\}^{r'}_{i=1}$ are two ensembles generating $\rho$, then there exists a unitary $U$ such that

$$\sum_{i=1}^{r} \sqrt{p_i} |\psi_i\rangle^{AB} |i\rangle^C = \sum_{i=1}^{r'} \sqrt{p'_i} |\psi'_i\rangle^{AB} U |i\rangle^C$$

$$= \sum_{i=1}^{r'} \sqrt{p'_i} |\psi'_i\rangle^{AB} |i'\rangle^C,$$

where $U |i\rangle = |i'\rangle$ is a change in basis of system $C$.

So if Charlie measures in the $\{|i\rangle\}$ basis he prepares the ensemble $\mathcal{E}_1$ for Alice and Bob. If he measures in the $\{|i'\rangle\}$ basis he prepares the ensemble $\mathcal{E}_2$ for Alice and Bob.
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In general, to say that Charlie “prepares the ensemble” $\mathcal{E} = \{ |\varphi_i\rangle, q_i \}$ for Alice and Bob means that Charlie makes a measurement on his system, and with probability $q_i$, Alice and Bob’s post-measurement state is $|\varphi_i\rangle$.

(Of course Alice and Bob know their post-measurement state only if Charlie tells them his measurement outcome)

If Charlie holds a purification of $\rho$, then he can prepare for Alice and Bob any pure-state ensemble that generates $\rho$.

Physical Interpretation of $\widehat{\mu}(\rho)$

If Alice and Bob are entangled with some Charlie in the pure state $|\omega\rangle^{ABC}$, then Alice and Bob’s average value of $\mu$ will be at least $\widehat{\mu}(tr_C |\omega\rangle\langle\omega|^{ABC})$ regardless of what measurement Charlie makes on his system.
Entanglement Entropy

The von Neumann entropy of a density matrix $\rho$ is defined by

$$S(\rho) = -\text{tr}[\rho \log \rho] = H(\{\lambda_i\}),$$

where the $\lambda_i$ are the eigenvalues of $\rho$.

Let $|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i} |\alpha_i\rangle |\beta_i\rangle$ be a Schmidt decomposition of $|\Psi\rangle$. The Entanglement Entropy of $|\Psi\rangle$ is defined as

$$E(|\Psi\rangle) = S(\text{tr}_A(|\Psi\rangle\langle\Psi|)) = H(\{\lambda_i\}).$$

The Entanglement of Formation of a mixed state $\rho$ is the convex roof extension of the Entanglement Entropy:

$$E_F(\rho) = \min_{\{p_i,|\varphi_i\rangle\}} \sum_i p_i E(|\varphi_i\rangle).$$
Entanglement Entropy

Example

Compute the Entanglement Entropy of the two-qubit state
\[ |\psi\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle. \]
Compute the von Neumann entropy of \( |\psi\rangle \).

Example

Compute the Entanglement Entropy of the two-qubit state
\[ |\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle. \]
Entanglement Entropy in Two Qubits

In the last example, we showed that

$$E(\psi) = h(\lambda),$$

where $h(x) = -x \log x - (1 - x) \log(1 - x)$ and

$$\lambda = \frac{1}{2} (1 + \sqrt{1 - 4|\text{det}(\rho^A)|}) = \frac{1}{2} (1 + \sqrt{1 - 4|ad - bc|^2}).$$

Recall the **magic basis** for two-qubit state space:

$$|e_1\rangle = \sqrt{1/2}(|00\rangle + |11\rangle) = |\Phi^+\rangle$$

$$|e_2\rangle = i\sqrt{1/2}(|00\rangle - |11\rangle) = i|\Phi^-\rangle$$

$$|e_3\rangle = i\sqrt{1/2}(|01\rangle + |10\rangle) = i|\Psi^+\rangle$$

$$|e_4\rangle = \sqrt{1/2}(|01\rangle - |10\rangle) = |\Psi^-\rangle.$$
Entanglement Entropy and Concurrence in Two Qubits

The concurrence of a bipartite two-qubit state $|\psi\rangle = \sum_{i=1}^{4} \alpha_k |e_k\rangle$ is defined as

$$C(\psi) = |\sum_{k=1}^{4} \alpha_k^2|,$$

where $\alpha_k$ are the components of $|\psi\rangle$ when written in the magic basis.

The coordinate transformation between the magic basis and the computational basis is given by

$$a = \frac{1}{\sqrt{2}} (\alpha_1 + i\alpha_2), \quad b = \frac{i}{\sqrt{2}} (\alpha_3 + i\alpha_4),$$
$$c = \frac{i}{\sqrt{2}} (\alpha_3 - i\alpha_4), \quad d = \frac{1}{\sqrt{2}} (\alpha_1 - i\alpha_2).$$
Magically, we see that

\[ C(\Psi) = 2|ad - bc| = 2\sqrt{|\det(\rho_A)|}. \]

Therefore, we can express the entanglement entropy of two qubits as a function of the concurrence:

\[ E(\Psi) = h \left( \frac{1}{2}(1 + \sqrt{1 - C(\Psi)^2}) \right). \]

Note that we can also express the concurrence as

\[ C(\Psi) = |\langle \Psi | \sigma_y \otimes \sigma_y | \Psi^* \rangle|, \]

where complex conjugation is taken in the computational basis.