The Partial Trace and the Schmidt Decomposition

The partial trace of any rank-one positive operator $|\psi\rangle\langle\psi|$ can easily be calculated in terms of a Schmidt decomposition of $|\psi\rangle$.

A Schmidt decomposition of $|\psi\rangle$ has the form

$$|\psi\rangle = \sum_{i=1}^{r} \sigma_i |\alpha_i\rangle \otimes |\beta_i\rangle$$

where the $|\alpha_i\rangle$ are orthonormal as well as the $|\beta_i\rangle$.

Then

$$\text{Tr}_A |\psi\rangle\langle\psi| = \sum_{i=1}^{r} \sigma_i^2 |\beta_i\rangle\langle\beta_i|$$

$$\text{Tr}_B |\psi\rangle\langle\psi| = \sum_{i=1}^{r} \sigma_i^2 |\alpha_i\rangle\langle\alpha_i|.$$
The Stern-Gerlach Experiment
The Axioms of Quantum Mechanics

**Axiom 1: The State Space Axiom**

Every quantum system is represented by a Hilbert space $\mathcal{H}$ called **state space**. Physical states of the system are in a one-to-one correspondence with rays in the Hilbert space.

**Rays** in a vector space are simply one-dimensional subspaces:

$$\{ \alpha |\psi\rangle : \alpha \in \mathbb{C} \},$$

where $|\psi\rangle$ is some unit vector.

The convention is to represent every physical state by a unit vector $|\psi\rangle$. Multiplying $|\psi\rangle$ by an **overall** phase $e^{i\phi}$ does not change any of the experimental outcomes predicted in quantum mechanics.

For a state $|\psi\rangle$ decomposed in a linear combination $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$, a **relative phase** is a factor $e^{i\phi}$ that is multiplied to just one of the kets:

$$|\psi'\rangle = \alpha |0\rangle + \beta e^{i\phi} |1\rangle.$$ Kets $|\psi\rangle$ and $|\psi'\rangle$ represent different physical states.
Axiom 2: The Unitary Evolution Axiom
Every closed quantum system evolves unitarily in time.

The axiom says that there exists a unitary operator $U(t_0, t_1)$ acting on $\mathcal{H}$ such that if the system is in state $|\psi\rangle$ at time $t_0$, it will be in state $U(t_0, t_1)|\psi\rangle$ at time $t_1$.

Since the evolution is unitary, $U(t_0, t_1)|\psi\rangle$ is still a unit vector.

Note the entire evolution can be reversed by applying the unitary $U(t_0, t_1)\dagger$. 
The Axioms of Quantum Mechanics

Axiom 3: The Measurement Axiom

A measurement on system \( \mathcal{H} \) is represented by a hermitian operator \( X \in \mathbb{L}(\mathcal{H}) \) called an observable, and conversely, every hermitian operator \( X \in \mathbb{L}(\mathcal{H}) \) corresponds to a physical measurement on \( \mathcal{H} \).

For an observable \( X \) with spectral decomposition \( X = \sum_{k=1}^{n} \lambda_k P_{\lambda_k} \), its eigenvalues \( \lambda_k \) are the different values that can be measured when performing the measurement described by \( X \). If \( |\psi\rangle \in \mathcal{H} \) is the pre-measurement state of the system, then value \( \lambda_k \) will be measured with probability

\[
p(\lambda_k) = \langle \psi | P_{\lambda_k} | \psi \rangle.
\]

When \( \lambda_k \) is measured, the post-measurement state of the system is

\[
\frac{1}{\sqrt{p(\lambda_k)}} P_{\lambda_k} | \psi \rangle.
\]
The Axioms of Quantum Mechanics

Quantum measurement is a stochastic process. We can only assign a probability distribution \( \{ p(\lambda_k) \}_{k=1}^n \) to the \( n \) different outcomes of a quantum measurement.

Conditioned on outcome \( \lambda_k \), the system undergoes the transformation

\[
\text{Pre-measurement: } |\psi\rangle \quad \rightarrow \quad \text{Post-measurement: } \frac{1}{\sqrt{p(\lambda_k)}} P_{\lambda_k} |\psi\rangle.
\]

The state of the system is projected into the \( \lambda_k \) eigenspace of observable \( X \) whenever \( \lambda_k \) is measured.
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States represented by eigenvectors of an observable are called **eigenstates**.

If \( |\lambda_k\rangle \) is any eigenvector in the eigenspace \( V_{\lambda_k} \) of \( X \), then

\[
P_{\lambda_l}|\lambda_k\rangle = \delta_{lk}|\lambda_k\rangle.
\]

This says that if \( l \neq k \), there is zero probability of obtaining outcome \( \lambda_l \) when the system is prepared in state \( |\lambda_k\rangle \).

The only possible outcome is \( \lambda_k \), and the state \( |\lambda_k\rangle \) remains unchanged in the measurement process. For this reason, eigenstates of an observable are referred to as **stationary states**.
The Axioms of Quantum Mechanics

Stern-Gerlach Example
Axiom 4: The Composite System Axiom

If $A$ and $B$ are two quantum systems with state spaces $\mathcal{H}^A$ and $\mathcal{H}^B$ respectively, the state space of the their combined physical system is the tensor product space $\mathcal{H}^{AB} := \mathcal{H}^A \otimes \mathcal{H}^B$. For more systems, $\mathcal{H}^{A_1}, \mathcal{H}^{A_2}, \ldots, \mathcal{H}^{A_n}$, the joint state space is the $n$-fold tensor product space $\mathcal{H}^{A_1} \otimes \mathcal{H}^{A_2} \otimes \cdots \otimes \mathcal{H}^{A_n}$. Two quantum systems are often called a bipartite system while more than two systems are typically referred to as a multipartite system.

A bipartite state $|\psi\rangle^{AB}$ is **entangled** if it is not a tensor product state; i.e. $|\psi\rangle^{AB} \neq |\alpha\rangle^A |\beta\rangle^B$. A non-entangled state is also called a **product** or **separable** state.
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How to decide if a state is entangled?

**Proposition:** A bipartite state $|\psi\rangle$ is a product state iff its operator representation $M|\psi\rangle$ is rank one.

Recall a nonzero matrix is rank 1 iff all of its $2 \times 2$-minors vanish.

**Example**

Decide if the following $\mathbb{C}^3 \otimes \mathbb{C}^3$ state is entangled:

$$|\psi\rangle^{AB} = a|11\rangle + \sqrt{ab}|12\rangle + c|13\rangle + \sqrt{ab}|21\rangle + b|22\rangle + c\sqrt{b/a}|23\rangle - c|33\rangle.$$