Homework 1

February 5, 2017

(1)

Let $|\psi\rangle = \frac{i}{\sqrt{3}}|00\rangle - \sqrt{\frac{2}{3}}|11\rangle$ be a vector in a four-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^4$. Find three vectors $|\varphi_1\rangle$, $|\varphi_2\rangle$ and $|\varphi_3\rangle$ such that the set $\{|\psi\rangle, |\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle\}$ forms an orthonormal basis for $\mathcal{H}$.

**SOLUTION:**

Our task is to construct three normalized vectors $\{|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle\}$ that are pairwise orthogonal to each other and orthogonal to $|\psi\rangle$. There are an infinite number of possibilities here, but let’s make it as easy as possible. Clear the vectors $|\varphi_1\rangle = |01\rangle$ and $|\varphi_2\rangle = |10\rangle$ satisfy the required properties. In order for the third state to be orthogonal to both of these, it must have the form $|\varphi_3\rangle = \alpha|00\rangle + \beta|11\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$. To be orthogonal to $|\psi\rangle$, we need $-i\alpha + \sqrt{2}\beta = 0$. That is, we need $\alpha = i\sqrt{2}\beta$.

Substituting this into the normalization condition gives $2|\beta|^2 + |\beta|^2 = 1$. Thus, we have a solution $|\varphi_3\rangle = i\sqrt{\frac{2}{3}}|00\rangle + \sqrt{\frac{1}{3}}|11\rangle$.

(2)

Consider the projector $|\psi\rangle\langle\psi|$ onto the one-dimensional subspace spanned by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$, where $\alpha, \beta, \gamma \in \mathbb{C}$ and satisfy the normalization condition $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. Write the general expression for the resulting vector after applying $|\psi\rangle\langle\psi|$ to the state $|\varphi\rangle = \sqrt{1/3}(|0\rangle + |1\rangle + |2\rangle)$. Is this vector normalized?

**SOLUTION:**

It is

$$\langle\psi\rangle\langle\psi| |\varphi\rangle = |\psi\rangle\langle\psi| |\varphi\rangle \quad \quad = \sqrt{1/3}(\alpha^* + \beta^* + \gamma^*)|\psi\rangle. \quad \quad (1)$$

The vector is not normalized.

(3)

Prove the following properties directly from the definition of the adjoint.

(a) For operators $A$ and $B$ and $a, b \in \mathbb{C}$, $(aA + bB)\dagger = a^*A\dagger + b^*B\dagger$;

(b) For operators $A$ and $B$, $(AB)\dagger = B\dagger A\dagger$;

(c) For vectors $|\psi\rangle$ and $|\varphi\rangle$, $(|\psi\rangle\langle\varphi|)\dagger = |\varphi\rangle\langle\psi|$. 

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SOLUTION:

(a) Let $|v\rangle$ and $|w\rangle$ be arbitrary vectors. Then

$$
\langle v|(aA + bB)^\dagger|w\rangle^* = \langle w|aA + bB|v\rangle
$$

$$
= a\langle w|A|v\rangle + b\langle w|B|v\rangle,
$$

where the first equality follows from the definition of the adjoint, and the second inequality follows from properties of the inner product. Now take the complex conjugate of both sides to obtain

$$
\langle v|(aA + bB)^\dagger|w\rangle^* = a^*\langle w|A|v\rangle^* + b^*\langle w|B|v\rangle^*
$$

$$
= a^*\langle v|A^\dagger|w\rangle + b^*\langle v|B^\dagger|w\rangle
$$

$$
= \langle v|a^*A^\dagger + b^*B^\dagger|w\rangle.
$$

Since this holds for arbitrary $|v\rangle$ and $|w\rangle$, we must have the operator equality $(aA + bB)^\dagger = a^*A^\dagger + b^*B^\dagger$.

(b) For arbitrary $|v\rangle$, let $|v'\rangle = B|v\rangle$. What is the bra of $|v'\rangle$? It is given by $\langle v|B^\dagger$. To prove this, note that for any $|w\rangle$ we have

$$
\langle v'|w\rangle = \langle w|v'\rangle^* = \langle w|B|v\rangle^* = \langle v|B^\dagger|w\rangle = (\langle v|B^\dagger \rangle|w\rangle).
$$

Since this holds for any $|w\rangle$, we have established that $\langle v'| = \langle v|B^\dagger$. Now consider the product $AB$. We have

$$
\langle v|(AB)^\dagger|w\rangle = \langle w|AB|v\rangle^* = \langle w|A|v'\rangle^* = \langle v'|A^\dagger|w\rangle = \langle v|B^\dagger A^\dagger|w\rangle.
$$

Hence, $(AB)^\dagger = B^\dagger A^\dagger$.

(c) For arbitrary $|v\rangle$ and $|w\rangle$, we have

$$
\langle v|(|\psi\rangle\langle\phi|)^\dagger|w\rangle = \langle w|(|\psi\rangle\langle\phi|)|v\rangle^*
$$

$$
= (\langle w|\psi\rangle\langle\phi|v\rangle)^*
$$

$$
= \langle w|\psi\rangle^*\langle\phi|v\rangle^*
$$

$$
= \langle \psi|w\rangle \langle v|\phi\rangle = \langle v|\phi\rangle \langle \psi|w\rangle
$$

$$
= \langle v|\phi\rangle \langle \psi|w\rangle = \langle v|\phi\rangle \langle \psi|w\rangle = \langle v|\phi\rangle \langle \psi|w\rangle.
$$

Hence, $(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi|$.


(4)

(a) In this exercise, you will show that the conjugate transpose is a basis-independent operation. For an arbitrary operator $M$ acting on Hilbert space $\mathcal{H}$, suppose that $M_\alpha$ and $M_\beta$ are two different matrix representations of $M$ with respect to two different orthonormal bases $\{|\alpha_i\rangle\}_{i=1}^d$ and $\{|\beta_i\rangle\}_{i=1}^d$. Prove that $(M_\alpha^T)^T$ and $(M_\beta^T)^T$ are matrix representations for the same operator $M^\dagger$. [Hint: Use the properties you proved in Problem (3)]

(b) Show that, in contrast, the transpose is a basis-dependent operation. That is, find some operator $M$ having matrix representations $M_\alpha$ and $M_\beta$ in two different bases such that $M_\alpha^T$ and $M_\beta^T$ are not matrix representations for the same operator.
SOLUTION:

(a) Let $N$ be the operator whose representation in the $\alpha$ basis is $(M_{\alpha}^*)^T$. Thus

$$[[M_{\alpha}^*]]_{i,j} = \langle \alpha_i | N | \alpha_j \rangle \quad \Rightarrow \quad [[[M_{\alpha}]]]_{j,i} = \langle \alpha_i | N | \alpha_j \rangle^* = \langle \alpha_j | N^\dagger | \alpha_i \rangle.$$ 

This shows that $N^\dagger = M$, and hence $(M_{\alpha}^*)^T$ is the matrix representation for $M^\dagger$. An analogous argument shows that $(M_{\beta}^*)^T$ is also a matrix representation for $M^\dagger$.

(b) Let us consider the operator $\sigma_y$ whose action on the computational basis is $\sigma_y |0\rangle = i |1\rangle$ and $\sigma_y |1\rangle = -i |0\rangle$. In the computational basis (denote this by the “$\alpha$” basis), $\sigma_y$ has the representation

$$\sigma_y \doteq M_\alpha := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$ 

It can be easily compute that $\sigma_y$ has eigenvalues $\pm 1$ with respective eigenvectors $|\pm\rangle := \sqrt{1/2} (|0\rangle \pm i |1\rangle)$. Hence in the $\{ |\pm\rangle, |\mp\rangle \}$ basis (denote this by the “$\beta$” basis), $\sigma_y$ has the matrix representation

$$\sigma_y \doteq M_\beta := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Since $M_\alpha^T \neq M_\alpha$, we have that $M_\alpha^T$ does not represent $\sigma_y$ in the computational basis, but in contrast $M_\beta^T$ does represent $\sigma_y$ in the $|\pm\rangle$ basis since $M_\beta = M_\beta^T$. Therefore $M_\alpha^T$ and $M_\beta^T$ are matrix representations of two different operators.

(5)

For an operator $A$, show that its trace $Tr(A)$ is independent of the basis in which its expressed.

SOLUTION:

Let $\{ |a_i\rangle \}_{i=1}^d$ and $\{ |b_j\rangle \}_{j=1}^d$ be any two orthonormal bases for $\mathcal{H}$. To prove that the trace is basis-independent, we will use the fact that $I = \sum_{i=1}^d |a_i\rangle \langle a_i| = \sum_{j=1}^d |b_j\rangle \langle b_j|$. Then

$$\sum_{i=1}^d \langle a_i | A | a_i \rangle = \sum_{i=1}^d \langle a_i | [A] | a_i \rangle$$

$$= \sum_{i=1}^d \sum_{j,k=1}^d \langle a_i | (|b_j\rangle \langle b_j| A |b_k\rangle \langle b_k|) |a_i \rangle$$

$$= \sum_{i=1}^d \sum_{j,k=1}^d \langle a_i | b_j \rangle \langle b_j | A | b_k \rangle \langle b_k | a_i \rangle$$

$$= \sum_{i=1}^d \sum_{j,k=1}^d \langle b_j | A | b_k \rangle \langle b_k | a_i \rangle \langle a_i | b_j \rangle$$

$$= \sum_{j,k=1}^d \langle b_j | A | b_k \rangle \langle b_k | b_j \rangle = \sum_{j=1}^d \langle b_j | A | b_j \rangle.$$ (6)

Hence, $Tr(A) = \sum_{i=1}^d \langle a_i | A | a_i \rangle = \sum_{j=1}^d \langle b_j | A | b_j \rangle$ is a basis-independent number.
(a) Consider the operator $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ expressed in the computational basis. Write out the matrix representation for $\sigma_y \otimes \sigma_y$ in the computational basis and compute its eigenvalues/eigenvectors.

(b) For the states $|\psi\rangle$ and $|\varphi\rangle$ given in Problem (2), write out the matrix representation of $|\psi\rangle\langle\varphi|$ in the computational basis and compute its eigenvalue/eigenvectors.

**SOLUTION:**

(a) In the computational basis, 

$$\sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$ 

(7)

To compute the eigenvalues/eigenvectors, note that $\sigma_y$ has eigenvectors $|\tilde{\pm}\rangle := \sqrt{1/2}(|0\rangle \pm i|1\rangle)$ with corresponding eigenvalues $\pm 1$. That is, $\sigma_y|\tilde{\pm}\rangle = \pm|\tilde{\pm}\rangle$. This means that when moving to the tensor product, we have 

$$\sigma_y \otimes \sigma_y|\tilde{+}\rangle \otimes |\tilde{+}\rangle = 1, \quad \sigma_y \otimes \sigma_y|\tilde{-}\rangle \otimes |\tilde{-}\rangle = 1$$

$$\sigma_y \otimes \sigma_y|\tilde{+}\rangle \otimes |\tilde{-}\rangle = -1, \quad \sigma_y \otimes \sigma_y|\tilde{-}\rangle \otimes |\tilde{+}\rangle = -1.$$ 

(8)

Therefore, $\sigma_y \otimes \sigma_y$ has eigenvalues $\{+1, -1\}$ and each of these have a degeneracy of 2. The projectors onto the $\pm 1$ eigenspaces are 

$$P_+ = |\tilde{+}\rangle \langle \tilde{+}| + |\tilde{-}\rangle \langle \tilde{-}|$$

$$P_- = |\tilde{-}\rangle \langle \tilde{+}| + |\tilde{+}\rangle \langle \tilde{-}|.$$ 

(9)

(b) 

$$|\psi\rangle\langle\varphi| = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & \beta & \beta \\ \gamma & \gamma & \gamma \end{pmatrix} = \sqrt{1/3} \begin{pmatrix} \alpha & \alpha & \alpha \\ \beta & \beta & \beta \\ \gamma & \gamma & \gamma \end{pmatrix}.$$ 

(10)

Notice that 

$$\langle|\psi\rangle\langle\varphi||\alpha\rangle = \langle\varphi|\alpha|\psi\rangle = \lambda|\psi\rangle \text{ for } \lambda \neq 0 \text{ if } \langle\varphi|\alpha\rangle \neq 0.$$ 

Hence $\langle\varphi|\psi\rangle$ is an eigenvalue of $|\psi\rangle\langle\varphi|$ with associated eigenspace being the one-dimensional space spanned by $|\psi\rangle$. The other eigenvalue is 0 with associated eigenspace being $\{|--\rangle\}$, the orthogonal complement of the one-dimensional space spanned by $|\varphi\rangle$.

(7)

Suppose a vector $|\psi\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B$ has a Schmidt decomposition $|\psi\rangle = \sum_{i=1}^r \sqrt{p_i}|\alpha_i\rangle^A|\beta_i\rangle^B$. What is the resulting vector after applying the operator $1 \otimes |\beta_k\rangle\langle\beta_k|$ to $|\psi\rangle$ for any $1 \leq k \leq r$, where $\mathbb{I}$ is the identity operator on $\mathcal{H}^A$?

**SOLUTION:**
Compute

\[(I \otimes |\beta_k\rangle\langle\beta_k|)|\psi\rangle = \sum_{i=1}^{r} \sqrt{p_i}|\alpha_i\rangle|\beta_k\rangle\langle\beta_k|\beta_i\rangle = \sqrt{p_k}|\alpha_k\rangle|\beta_k\rangle,\]  

(11)

where we are using the fact that the Schmidt vectors \(|\beta_k\rangle\} forms an orthonormal basis.