Homework 2

(1)
Prove that any two antipodal points on the Bloch sphere correspond to orthogonal quantum states.

**SOLUTION:**

Let \( \hat{x} = (x, y, z) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) be any point on the Bloch sphere. Its antipodal point is the only other unit vector parallel to \((x, y, z)\); hence the coordinates of the antipodal point must be \(-\hat{x} = -(x, y, z) = (\sin(\theta + \pi) \cos \phi, \sin(\theta + \pi) \sin \phi, \cos(\theta + \pi))\). The corresponding quantum states are 
\( |\hat{x}\rangle = \cos(\theta / 2)|0\rangle + \sin(\theta / 2)e^{i\phi}|1\rangle \) and 
\( |-\hat{x}\rangle = \cos(\theta / 2 + \pi / 2)|0\rangle + \sin(\theta / 2 + \pi / 2)e^{-i\phi}|1\rangle \). It is straightforward to now check that \(0 = \langle \hat{x} | -\hat{x} \rangle\).

(2)
For arbitrary unit vectors \( \hat{r} \) and \( \hat{n} \) in \( \mathbb{R}^3 \), prove the following relationships

(a) \((\hat{r} \cdot \hat{\sigma})(\hat{n} \cdot \hat{\sigma}) = (\hat{n} \cdot \hat{r})\mathbf{1} + i(\hat{r} \times \hat{n}) \cdot \hat{\sigma}\); 

(b) \((\hat{n} \cdot \hat{\sigma})(\hat{r} \cdot \hat{\sigma})(\hat{n} \cdot \hat{r}) = 2(\hat{n} \cdot \hat{\sigma})(\hat{n} \cdot \hat{r}) - (\hat{r} \cdot \hat{\sigma})\).

**SOLUTION:**

(a) For vectors \( \hat{r} = (r_1, r_2, r_3) \) and \( \hat{n} = (n_1, n_2, n_3) \), we compute 
\[
(\hat{r} \cdot \hat{\sigma})(\hat{n} \cdot \hat{\sigma}) = \sum_{j,k} r_j n_k \sigma_j \sigma_k = \sum_{j,k} r_j n_k (\delta_{jk} \mathbf{1} + i \epsilon_{jkl} \sigma_l) = (\hat{n} \cdot \hat{r})\mathbf{1} + i \sum_{j,k} r_j n_k \epsilon_{jkl}
\]
\[
= (\hat{n} \cdot \hat{r})\mathbf{1} + i(\hat{r} \times \hat{n}) \cdot \hat{\sigma}.
\]

For (b), we use (a) to obtain 
\[
(\hat{n} \cdot \hat{\sigma})(\hat{r} \cdot \hat{\sigma})(\hat{n} \cdot \hat{r}) = (\hat{n} \cdot \hat{\sigma})[(\hat{n} \cdot \hat{r})\mathbf{1} + i(\hat{r} \times \hat{n}) \cdot \hat{\sigma}]
\]
\[
= (\hat{n} \cdot \hat{\sigma})(\hat{n} \cdot \hat{r}) + i(\hat{n} \cdot \hat{\sigma})(\hat{r} \times \hat{n}) \cdot \hat{\sigma}
\]
\[
= (\hat{n} \cdot \hat{\sigma})(\hat{n} \cdot \hat{r}) + i[\hat{n} \cdot (\hat{r} \times \hat{n})\mathbf{1} + i\hat{n} \times (\hat{r} \times \hat{n}) \cdot \hat{\sigma}]
\]
\[
= (\hat{n} \cdot \hat{\sigma})(\hat{n} \cdot \hat{r}) - \hat{r} \cdot \hat{\sigma} + (\hat{n} \cdot \hat{\sigma})(\hat{n} \cdot \hat{r}) = 2(\hat{n} \cdot \hat{\sigma})(\hat{n} \cdot \hat{r}) - (\hat{r} \cdot \hat{\sigma}).
\]
(a) Show that every element $U$ of $SU(2)$ can be written as

$$U = \pm R_{\hat{n}}(\theta) = \pm e^{i\hat{n} \cdot \hat{\sigma}/2}.$$ 

(b) Consider the Hadamard gate $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Show that there exists an overall phase $e^{i\gamma}$ so that $H' := e^{i\gamma}H \in SU(2)$. What are the values of $\theta$ and $\hat{n}$ for $H'$?

(c) Consider the complex phase gate $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. Show that there exists an overall phase $e^{i\gamma}$ so that $S' := e^{i\gamma}S \in SU(2)$. What are the values of $\theta$ and $\hat{n}$ for $S'$?

**SOLUTION:**

(a) Any $2 \times 2$ complex matrix can be written as a linear combination of the elements $\{I, \sigma_x, \sigma_y, \sigma_z\}$. That is, we can write $U = e^{i\gamma}(aI + \vec{r} \cdot \hat{\sigma})$, where $a$ is a real number and $r_x, r_y, r_z$ are complex numbers. Unitarity implies that

$$I = (aI + \vec{r} \cdot \hat{\sigma})(aI + \vec{r}^* \cdot \hat{\sigma}) = |a|^2I + \sum_l 2Re(ar_l)\sigma_l + \sum_{j,k} r_j r_k^* \sigma_j \sigma_k$$

$$= (a^2 + |\vec{r}|^2)I + \sum_l 2aRe(r_l)\sigma_l + i \sum_{jkl} e_{jkl} r_j r_k^* \sigma_l$$

$$I = (aI + \vec{r}^* \cdot \hat{\sigma})(aI + \vec{r} \cdot \hat{\sigma}) = (a^2 + |\vec{r}|^2)I + \sum_l 2aRe(r_l)\sigma_l + i \sum_{jkl} e_{jkl} r_j^* r_k \sigma_l$$

Adding these together gives $2I = 2(a^2 + |\vec{r}|^2)I + 2 \sum_l 2aRe(r_l)\sigma_l$, since $\sum_{jkl} e_{jkl} (r_j r_k^* + r_j^* r_k) = 0$. Thus, we must have $Re(r_l) = 0$ for all $l$ and $a^2 + |\vec{r}|^2 = 1$. In other words, $\vec{r}$ is a has purely imaginary components. After normalizing $\vec{r}$, we see that $U$ has the form

$$U = e^{i\gamma}(\cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} \hat{n} \cdot \hat{\sigma}) = e^{i\gamma} \left( \cos \frac{n_y - in_x}{2} \sin \frac{n_z}{2} \left( -n_y - in_x \right) \cos \frac{n_z}{2} + \sin \frac{n_z}{2} \right),$$

where $\hat{n}$ is a real unit vector. The determinant is computed to be $e^{2i\gamma}$ (recall that $\det(aM) = a^d \det(M)$ for any $d \times d$ matrix $M$ and $a \in \mathbb{C}$). Thus for $U$ to be in $SU(2)$, we need $e^{2i\gamma} = 1$, which means that $e^{i\gamma} = \pm 1$. Therefore $U = \pm (\cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} \hat{n} \cdot \hat{\sigma}) = \pm e^{i\hat{n} \cdot \hat{\sigma}/2}$.

(b) Since $\det(H) = -1$, we need $e^{2i\gamma} = -1$ in order for $H' \in SU(2)$. Thus, take $\gamma = \pi/4$. Now we need values of $\theta$ and $\hat{n}$ such that $H' = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\pi/4} & e^{-i\pi/4} \\ e^{-i\pi/4} & e^{i\pi/4} \end{pmatrix} = (\cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} \hat{n} \cdot \hat{\sigma})$. Since $e^{i\pi/4} = \frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}$, we obtain $H'$ by choosing $\theta = 2\pi/3$ and $\hat{n} = -\left( \sqrt{1/3}, \sqrt{1/3}, \sqrt{1/3} \right)$, so that $n_i \sin \theta = -\sqrt{1/2}$ for $i = x, y, z$.

(c) Since $\det(S) = i$, we need $e^{i2\gamma} = -i$ in order for $S' \in SU(2)$. Thus, take $\gamma = 3\pi/4$. We need values of $\theta$ and $\hat{n}$ such that $S' = \begin{pmatrix} e^{i3\pi/4} & 0 \\ 0 & e^{-i3\pi/4} \end{pmatrix}$. We obtain $S'$ by choosing $\theta = 3\pi/2$ and $\hat{n} = -(0, 0, 1)$. 


Consider a bipartite system $\mathbb{C}^2 \otimes \mathbb{C}^2$ initially in the state $|\Psi\rangle^{AB} = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$. Suppose that a von Neumann measurement is performed on system $B$ given by the eigenstates of $\sigma_y$. What is the probability distribution of measurement outcomes, and what are the possible post-measurement states of system $A$?

**SOLUTION:**

The eigenstates of $\sigma_y$ are $|\pm\rangle = \sqrt{1/2}(|0\rangle \pm i|1\rangle)$. To compute the post-measurement state on system $A$, we first compute the partial contraction

$$B(\pm) \left( |\Psi\rangle^{AB} \right) = \frac{1}{\sqrt{2}} (a|0\rangle \pm b|0\rangle + c|1\rangle \pm d|1\rangle).$$

The probabilities of the two outcomes are given by the norms of these vectors, which are $p(\pm) = \frac{1}{2}(|a \pm b|^2 + |c \pm d|^2)$. Thus, the two post-measurement states are

$$|\psi\rangle^B = \frac{1}{\sqrt{|a \pm b|^2 + |c \pm d|^2}} [(a \pm b)|0\rangle + (c \pm d)|1\rangle].$$

For any integer $N$, let $U = e^{-i\pi \sigma_y / N}$ be a unitary operator defined on $\mathbb{C}^2$.

(a) Write out the matrix representation of $U^k$ for any $k = 0, \ldots, N - 1$.

(b) Define the operators $M_k = \gamma U^k|0\rangle \langle 0| (U^k)^\dagger$ for $k = 0, \cdots, N_1$ and some constant $\gamma$. Show that for a suitable choice of $\gamma$, the operators $\{M_k\}_{k=0}^{N-1}$ represent a generalized measurement on $\mathbb{C}^2$. What is the value of $\gamma$?

**SOLUTION:**

For (a) we perform a Taylor expansion:

$$U^k = e^{-ik\pi \sigma_y / N} = \sum_{m=0}^{\infty} \frac{1}{m!} \left( -\frac{k\pi}{N} \right)^m \sigma_y^m = \sum_{m \text{ even}} \frac{1}{m!} \left( \frac{k\pi}{N} \right)^m - \sum_{m \text{ odd}} \frac{1}{m!} \left( \frac{k\pi}{N} \right)^m = \mathbb{1} \cos\left( \frac{k\pi}{N} \right) - i\sigma_y \sin\left( \frac{k\pi}{N} \right) = \begin{pmatrix} \cos(k\pi / N) & -\sin(k\pi / N) \\ \sin(k\pi / N) & \cos(k\pi / N) \end{pmatrix}. \quad (4)$$

For (b), we have that $U^k|0\rangle = \cos(k\pi / N)|0\rangle + \sin(k\pi / N)|1\rangle$. Then

$$\sum_{k=0}^{N-1} M_k^* M_k = |\gamma|^2 \sum_{k=0}^{N-1} \begin{pmatrix} \cos^2(k\pi / N) & \cos(k\pi / N) \sin(k\pi / N) \\ \cos(k\pi / N) \sin(k\pi / N) & \sin^2(k\pi / N) \end{pmatrix}.$$
Now, using the geometric series expansion described in the hint above, we just compute the sum over in each matrix element:

\[
\sum_{k=0}^{N-1} \cos(k\pi/N) = \sum_{k=0}^{N-1} \left( e^{ik\pi/N} + e^{-ik\pi/N} \right) = \frac{N}{2},
\]

\[
\sum_{k=0}^{N-1} \sin(k\pi/N) = \sum_{k=0}^{N-1} \left( e^{ik\pi/N} - e^{-ik\pi/N} \right) = 0,
\]

\[
\sum_{k=0}^{N-1} \sin^2(k\pi/N) = \sum_{k=0}^{N-1} \left( 1 - \cos^2(k\pi/N) \right) = N - \frac{N}{2} = \frac{N}{2}.
\]  

Hence the \( \{ M_k \}_{k=0}^{N-1} \) form a valid quantum measurement with \( \gamma = \sqrt{2/N} \).

(6)

Is the following state entangled

\[ |\Psi\rangle = ad|00\rangle + ae|02\rangle + be|12\rangle + cd|20\rangle + cd|22\rangle \] ?

**SOLUTION:**

To be a product state, all \( 2 \times 2 \) minors of the matrix

\[
M_{\Psi} = \begin{pmatrix}
ad & 0 & ae \\
bd & 0 & be \\
0d & 0 & cd
\end{pmatrix}
\]

must vanish. We’re left with needing to satisfy the equalities \( acd^2 = aecd \) and \( bcd^2 = becd \). Hence the state is entangled iff \( d \neq e \).

(7)

Prove the following statements.

(a) For any two product states \( |\psi_1\rangle = |\alpha_1\rangle|\beta_1\rangle \) and \( |\psi_2\rangle = |\alpha_2\rangle|\beta_2\rangle \) in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \), let \( S \) be the subspace spanned by \( |\psi_1\rangle \) and \( |\psi_2\rangle \). Then either all states in \( S \) are product states, or \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are the only product states in \( S \).

(b) Any two-dimensional subspace of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) contains at least one product state.

**SOLUTION:**

(a) Let \( |\psi_1\rangle = |00\rangle \), and expand \( |\psi_2\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \). The assumption that \( |\psi_2\rangle \) is a product state means that \( ad = bc \). Now, suppose there exists
some product state in the linear span of $|\psi_1\rangle$ and $|\psi_2\rangle$. That is, $x|\psi_1\rangle + y|\psi_2\rangle$ is a product state for some values of $x$ and $y$. In terms of the matrix representation, this means that $xM_{\psi_1} + yM_{\psi_2}$ is a rank one matrix, where $M_{\psi_i}$ is the matrix for $|\psi_i\rangle$. Since matrix rank is left unchanged when multiplying by an overall scalar, without loss of generality we can assume that $x = 1$. Then we have a product state iff

$$0 = \det (M_{\psi_1} + yM_{\psi_2}) = \det \begin{pmatrix} 1 + ya & yb \\ yc & yd \end{pmatrix} = yd + y^2 \det (M_{\psi_2}) = 0.$$  

Thus, a product state requires either $y = 0$ or $d = 0$. When $y = 0$, the product state is $|\psi_1\rangle$. When $d = 0$, then the condition $ad = bc$ requires either $b = 0$ or $c = 0$. Thus either $|\psi_2\rangle = (a|0\rangle + c|1\rangle)|0\rangle$ or $|\psi_2\rangle = (0)(a|0\rangle + b|1\rangle)$. In both cases, $|\psi_1\rangle + y|\psi_2\rangle$ will be product state for every choice of $y$. This completes the proof.

(b) Let $|\psi_1\rangle = \sqrt{\lambda_1}|00\rangle + \sqrt{\lambda_2}|11\rangle$ and $|\psi_2\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ be two arbitrary states. Then $|\psi_1\rangle + y|\psi_2\rangle$ will be an unnormalized product state iff

$$0 = \det \begin{pmatrix} \sqrt{\lambda_1} + ya & yb \\ yc & \sqrt{\lambda_2} + yd \end{pmatrix} = y^2 A + yB + C,$$

where $A = ad - bc$, $B = a\sqrt{\lambda_2} + d\sqrt{\lambda_1}$, and $C = \sqrt{\lambda_1\lambda_2}$. This is a quadratic equation in $y$, and it will always have at least one nonzero solution by the quadratic formula.

**Entanglement in the “Magic Basis” and Non-Entangling Gates.**

The **magic basis** for two-qubit state space is the orthonormal basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle, |e_4\rangle\}$ given by

$$|e_1\rangle = \sqrt{1/2}(|00\rangle + |11\rangle) = |\Phi^+\rangle$$

$$|e_2\rangle = i\sqrt{1/2}(|00\rangle - |11\rangle) = i|\Phi^-\rangle$$

$$|e_3\rangle = i\sqrt{1/2}(|01\rangle + |10\rangle) = i|\Psi^+\rangle$$

$$|e_4\rangle = \sqrt{1/2}(|01\rangle - |10\rangle) = |\Psi^-\rangle.$$  

The **concurrence** of a bipartite two-qubit state $|\Psi\rangle = \sum_{k=1}^4 a_k |e_k\rangle$ is defined as

$$C(\Psi) = \left| \sum_{k=1}^4 a_k^2 \right|,$$

where $a_k$ are the components of $|\Psi\rangle$ when written in the magic basis.

(8)

(a) Show that a given two-qubit state $|\Psi\rangle$ is entangled if and only $C(\Psi) = 0$.

(b) Show that a state $|\Psi\rangle$ has maximal concurrence if and only if it is real when expressed in the magic basis.
(c) What is the SWAP operator $\mathcal{F}$ in the magic basis? Show that $C(\Psi)$ remains invariant under the transformation $|\Psi\rangle \rightarrow \mathcal{F}|\Psi\rangle$.

**SOLUTION:**

(a) Let $|\Psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$. The relationship between the two bases is given by:

\[
|00\rangle = \sqrt{1/2}(|e_1\rangle - i|e_2\rangle), \quad |01\rangle = \sqrt{1/2}(-i|e_3\rangle + |e_4\rangle) \\
|10\rangle = \sqrt{1/2}(-i|e_3\rangle - |e_4\rangle), \quad |11\rangle = \sqrt{1/2}(|e_1\rangle + i|e_2\rangle). \tag{7}
\]

Thus

\[
a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle = \sqrt{1/2}[a(|e_1\rangle - i|e_2\rangle) + b(-i|e_3\rangle + |e_4\rangle) \\
+ c(-i|e_3\rangle - |e_4\rangle) + d(|e_1\rangle + i|e_2\rangle)] \\
= \sqrt{1/2}[(a + d)|e_1\rangle - i(a - d)|e_2\rangle \\
+ -i(b + c)|e_3\rangle + (b - c)|e_4\rangle]. \tag{8}
\]

The concurrence is computed as

\[
C(\Psi) = \frac{1}{2}[(a + d)^2 - (a - d)^2 - (b + c)^2 + (b - c)^2] \\
= |ad - bc|. \tag{9}
\]

Hence $|\Psi\rangle$ is entangled iff $C(\Psi) > 0$.

(b) The concurrence is upper bounded by

\[
C(\Psi) = \frac{1}{2} \sum_{k=1}^{4} \alpha_k^2 \leq \sum_{k=1}^{4} \alpha_k^2 \leq \sum_{k=1}^{4} |\alpha_k|^2 = 1,
\]

and equality is obtained iff each of the $\alpha_k$ is real.

(c) Using Eq. (7) we compute that

\[
\mathcal{F} = |00\rangle\langle 00| + |10\rangle\langle 10| + |01\rangle\langle 01| + |11\rangle\langle 11| = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + |e_3\rangle\langle e_3| - |e_4\rangle\langle e_4|.
\]

It is obvious that $C(\Psi)$ remains invariant under SWAP since each of the magic basis states are eigenvectors of $\mathcal{F}$ with real eigenvalues.

(9)

Prove that a two-qubit gate $U^{AB}$ is non-entangling if and only if (up to an overall phase factor) it is real in the magic basis.
SOLUTION:

The forward direction is fairly easy to prove. Suppose that $U = \sum_{jk} u_{jk} |e_j\rangle\langle e_k|$ is real; i.e. $\text{Im}[u_{jk}] = 0$. Let $|\Psi\rangle = \sum_{i=1}^4 \alpha_i |e_i\rangle$ be any unentangled state. Then $|\Psi\rangle = U|\Psi\rangle = \sum_{jk} u_{jk} \alpha_k |e_j\rangle$, and so

$$C(\Psi') = \left| \sum_j (\sum_k u_{jk} \alpha_k)^2 \right| = \left| \sum_j \sum_{l,k} u_{j,l} u_{l,j} \alpha_k \alpha_l \right| = \left| \sum_k \alpha_k^2 \right| = 0,$$

where we use the facts that $\sum_j u_{jk} u_{jl} = \delta_{jk}$ (since the $u_{jk}$ are real) and $C(\Psi) = 0$.

Conversely, suppose that $U = \sum_{jk} u_{jk} |e_j\rangle\langle e_k|$ is non-entangling. We use Proposition 1 which ensures the existence of operators $O_1$ and $O_2$ that are real and orthogonal in the computational basis such that $U = O_1^T \Lambda O_2$, where $\Lambda = \sum_{k=1}^2 e^{i\theta_k} |e_k\rangle\langle e_k|$ is complex diagonal. As just proven above, $O_1$ and $O_2^T$ are non-entangling since they are real in the magic basis. Therefore $U$ is non-entangling iff $\Lambda$ is non-entangling.

Let $|\Psi\rangle = \sum_{k=1}^2 \alpha_k |e_k\rangle$ be an arbitrary product state so that $\sum_{k=1}^2 \alpha_k^2 = 0$. Then $U|\Psi\rangle = \sum_{k=1}^2 e^{i\theta_k} \alpha_k |e_k\rangle$, and its concurrence is $\sum_{k=1}^2 e^{i\theta_k} \alpha_k^2$. Consider now product states with only two nonzero magic basis coefficients, $\alpha_m = \sqrt{1/2}$ and $\alpha_n = i\sqrt{1/2}$ for $m \neq n$. For $U|\Psi\rangle$ to be a product state we need $e^{i\theta_m} = e^{i\theta_n}$. As this holds for every $m$ and $n$, we have that $\Lambda = e^{i\theta} \mathbb{I}$ for some $\theta$. Thus, $U$ is real in the magic basis up to the overall phase $e^{i\theta}$.

**Proposition 1.** For any unitary matrix $U$, there exists real orthogonal matrices $O_1$ and $O_2$ such that $O_1 U O_2^T$ is a complex diagonal matrix $\Lambda$. Furthermore, all the diagonal elements of $\Lambda$ are phases $e^{i\theta_i}$ with $0 \leq \theta_i \leq \pi/2$.

**Proof.** We can always write $U = A + iB$ where $A$ and $B$ are real matrices. Let $A = LD_1^{1/2} R^T$ be a singular value decomposition of $A$, where $L$ and $R$ are real orthogonal matrices and $D_1^{1/2}$ is diagonal. Then $L U R^T = D_1^{1/2} + i\tilde{B}$, where $\tilde{B} = L B R^T$. Notice that $L U R^T$ is still unitary, and so we must have $\mathbb{I} = D + \tilde{B}^T B + i(\tilde{B} D_1^{1/2} - D_1^{1/2} \tilde{B}^T)$ and $\mathbb{I} = D + \tilde{B}^T \tilde{B} + i(D_1^{1/2} \tilde{B} - \tilde{B}^T D_1^{1/2})$. For the imaginary parts to vanish, $\tilde{B} D_1^{1/2} = D_1^{1/2} \tilde{B}^T$ and $D_1^{1/2} \tilde{B} = \tilde{B}^T D_1^{1/2}$. Thus, $\tilde{B} D = D \tilde{B}$. Letting $b_{ij}$ be the elements of $\tilde{B}$ and $d_{ij}$ the elements of $D$, this equation says

$$\sum_j b_{ij} d_{jk} \delta_{jk} = \sum_j \delta_{ij} d_{ij} b_{jk} \Rightarrow b_{jk} d_{kk} = b_{jk} d_{ii} \Rightarrow b_{jk} (d_{kk} - d_{ii}),$$

which implies that $\tilde{B}$ is block-diagonal with each block corresponding to a constant diagonal element of $D$. We perform a singular value decomposition on each of the blocks in $\tilde{B}$ by the block orthogonal matrices $\tilde{L}$ and $\tilde{R}$. Since $L D \tilde{R}^T = D$ by the block structure, we have that the real orthogonal matrices $O_1 = \tilde{L} \tilde{L}$ and $O_2 = \tilde{R} \tilde{R}$ provide simultaneous singular value decompositions of $A$ and $B$. This proves the first part of the proposition. For the second statement, note that since $O_1 U O_2^T = \Lambda$ is unitary, the diagonal elements of $\Lambda$ have unit norm; i.e. $\Lambda_{ii} = e^{i\theta_i}$. Further-
more since $e^{i\theta}$ is obtained from the singular value decompositions of $A$ and $B$, both its real and imaginary parts are nonnegative. Hence $0 \leq \theta \leq \pi/2$.

References