Homework 3

The Transpose and Partial Transpose.

(1) Let \{\langle 1 \rangle, \langle 2 \rangle, \ldots, \langle d \rangle \} be an orthonormal basis for \( \mathbb{C}^d \). The transpose map defined with respect to this basis is a superoperator \( \Gamma \) that acts on an operator \( M \in \mathcal{L}(\mathbb{C}^d) \) by

\[
M = \sum_{i,j=1}^{d} m_{ij} \langle i | \langle j | \rightarrow \ \Gamma(M) = \sum_{i,j=1}^{d} m_{ij} \langle j | \langle i |. \tag{1}
\]

As emphasized before, this is a basis-dependent operation in the sense that if \{\langle 1' \rangle, \langle 2' \rangle, \ldots, \langle d' \rangle \} is another orthonormal basis, then the transpose map defined with respect to this other basis is the superoperator \( \Gamma' \) with action given by

\[
M = \sum_{i,j=1}^{d} m'_{ij} \langle i' | \langle j' | \rightarrow \ \Gamma'(M) = \sum_{i,j=1}^{d} m'_{ij} \langle j' | \langle i' |. \tag{2}
\]

(a) By providing an explicit example, show that \( \Gamma(M) \neq \Gamma'(M) \) in general.

(b) Nevertheless, show that \( \Gamma(M) \) and \( \Gamma'(M) \) will always have the same eigenvalue spectrum.

(Hint. Part (a) is identical to a previous homework problem. But now do it again for good measure and to make sure you understand the point. For part (b), use the fact that any two orthonormal bases are related by a unitary transformation.)

(2) Now consider two systems, \( A \) and \( B \), with bipartite state space \( \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \) and computational product basis \( \{ \langle i | \langle j | \}_{{i,j=1}}^{d_A,d_B} \). The partial transpose map of system \( B \) with respect to the computational basis is the superoperator \( \mathcal{I}^A \otimes \Lambda^B \) acting on \( \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d) \), where \( \Lambda^B \) is the transpose map on \( B \) and \( \mathcal{I}^A \) is the trivial map on \( A \); i.e. \( \mathcal{I}^A(T) = T \) for all \( T \in \mathcal{L}(\mathbb{C}^d) \). For an arbitrary \( M \in \mathcal{L}(\mathbb{C}^d \otimes \mathbb{C}^d) \), the action of \( \mathcal{I}^A \otimes \Gamma^B \) is given by

\[
M = \sum_{i,k=1}^{d_A} \sum_{j,l=1}^{d_B} m_{ij,kl} \langle i | \langle k | \otimes \langle j | \langle l | \rightarrow \ \mathcal{I}^A \otimes \Gamma^B(M) = \sum_{i,k=1}^{d_A} \sum_{j,l=1}^{d_B} m_{ij,kl} \langle k | \langle l | \otimes \Gamma(\langle j | \langle l |^B) = \sum_{i,k=1}^{d_A} \sum_{j,l=1}^{d_B} m_{ij,kl} \langle i | \langle k | \otimes \langle l | \langle j |^B. \tag{3}
\]

The partial transpose map of system \( A \) with respect to the computational basis is similarly defined as the map \( \Gamma^A \otimes \mathcal{I}^B \). For a bipartite operator \( M \), its partial transpose of \( B \) is denoted as \( M_{\Gamma^B} := \mathcal{I}^A \otimes \Gamma^B(M) \), and its partial transpose of \( A \) is likewise denoted by \( M_{\Gamma^A} := \Gamma^A \otimes \mathcal{I}^B(M) \).
(a) Let $M$ be an operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Then it is represented as a $4 \times 4$ complex matrix in the computational basis. Give the matrix representations for both $M^\Gamma_A$ and $M^\Gamma_B$.

(b) For $\mathbb{C}^d \otimes \mathbb{C}^d$ state space, prove that taking a partial transpose does not affect the partial trace: 
\[ \text{Tr}_A(M) = \text{Tr}_A(M^\Gamma_A) \quad \text{and} \quad \text{Tr}_B(M) = \text{Tr}_B(M^\Gamma_B). \]

(c) If $M$ is hermitian, show that both $M^\Gamma_A$ and $M^\Gamma_B$ are also hermitian.

(d) If $M$ is hermitian, show that $M^\Gamma_A$ and $M^\Gamma_B$ have the same eigenvalue spectrum.

\textbf{(Hint)} For (d), suppose that $M^\Gamma_A|\psi\rangle = \lambda |\psi\rangle$. By part (c) we know that both $M^\Gamma_A$ and $M^\Gamma_B$ are hermitian and therefore $\lambda$ is real. Starting from the equality $M^\Gamma_A|\psi\rangle\langle\psi| = \lambda |\psi\rangle\langle\psi|$, take the full transpose of both sides to convert $M^\Gamma_A$ into $M^\Gamma_B$. Then take the conjugate transpose of both sides and use the fact that $M^\Gamma_B$ is hermitian to obtain eigenvalue relation for $M^\Gamma_B$. When performing these steps, note that $(|\psi\rangle\langle\psi|)^T = |\psi^*\rangle\langle\psi^*|$, where complex conjugation is taken with respect to the computation basis.

\textbf{Positive and Completely Positive Maps.}

(3) Consider the map $\Lambda(X) = \text{Tr}(X)I - X$ for $X \in \mathcal{L}(\mathbb{C}^d)$. Is $\Lambda$ a positive map? Is it completely positive?

(4) The fully dephasing map $\Delta$ is a completely positive superoperator that acts on a linear operator $X \in \mathcal{L}(\mathbb{C}^d)$ according to
\[ \Delta(X) = \sum_{i=1}^d |i\rangle\langle i|X|i\rangle\langle i|. \]  

(a) Restrict attention to $\mathbb{C}^3$, and consider the superoperator
\[ \Lambda(X) = 2 \left( \Delta(X) + \Delta(\Pi X \Pi^\dagger) \right) - X \quad \text{for} \quad X \in \mathcal{L}(\mathbb{C}^3), \]

where $\Pi = |1\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 1|$ is a permutation. Is $\Lambda$ completely positive?

(b)* Is $\Lambda$ given in Eq. (5) a positive map?

\textbf{(Hint)} For (a), consider $\langle \Phi_3^+ | \Lambda(\Phi_3^+) \otimes I | \Phi_3^+ \rangle$. For part (b), note that $\Lambda(X)$ will always be hermitian when $X \geq 0$. Then use Sylvester’s criterion to check for positivity of $\Lambda$.

\textbf{(Bibliography note)} The map in Eq. (5) was first presented by Choi in Ref. [Cho75].

\textbf{Entanglement Fidelity.}

For a quantum channel $\mathcal{E} : \mathcal{L}(\mathbb{C}^d) \rightarrow \mathcal{L}(\mathbb{C}^d)$, its \textbf{entanglement fidelity} is the quantity given by
\[ \langle \Phi_d^+ | I \otimes \mathcal{E}(\Phi_d^+) | \Phi_d^+ \rangle, \]
where $|\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle$ and $\Phi_d^+ = |\Phi_d^+\rangle\langle \Phi_d^+|$. The entanglement fidelity essentially measures how well entanglement is preserved when sending one half of the entangled state through the channel.
A Pauli channel is any qubit channel of the form
\[ \mathcal{E}(\rho) = (1 - p_1 - p_2 - p_3)\rho + p_1\sigma_x\rho\sigma_x + p_2\sigma_y\rho\sigma_y + p_3\sigma_z\rho\sigma_z \] (6)
for nonnegative numbers \( p_1, p_2, p_3 \) called channel parameters.

For a Pauli channel \( \mathcal{E} \), compute the density matrix \( \mathcal{I} \otimes \mathcal{E}(\Phi^+) \) and the entanglement fidelity as a function of the channel parameters \( p_1, p_2, p_3 \).

For any integer \( N \), consider again the unitary \( U = e^{-i\pi\sigma_y/N} \), and now define the qubit channel \( \mathcal{E}_N(\rho) = \frac{1}{N} \sum_{k=0}^{N-1} U^k \rho (U^k)^\dagger \). Compute the density matrix \( \mathcal{I} \otimes \mathcal{E}_N(\Phi^+) \) and the entanglement fidelity.

Teleportation Fidelity.

The standard teleportation fidelity of a bipartite state \( \rho \) is the quantity given by
\[ f(\rho) = \int d\hat{n} \langle \hat{n} | \Lambda_{T_0,\rho} (|\hat{n}\rangle \langle \hat{n}|) |\hat{n}\rangle, \] (7)
where \( \Lambda_{T_0,\rho} \) is the qubit channel induced by performing the standard teleportation protocol \( T_0 \) on the bipartite state \( \rho \).

As a function of the channel parameters \( p_1, p_2, p_3 \), compute the standard teleportation fidelity of the bipartite state obtained by sending half of the maximally entangled state through a Pauli channel. That is, compute the standard teleportation fidelity of the state \( \rho = \mathcal{I} \otimes \mathcal{E}(\Phi^+) \), where \( \mathcal{E} \) is the channel given by Eq. (6).

How well can Alice and Bob simulate the teleportation of a random qubit state using no shared entanglement? This is the question you will explore in this exercise.

(a) Consider first the scenario where Alice does not send Bob any classical information. She randomly generates a state \( |\hat{n}\rangle \) with \( \hat{n} \) distributed uniformly on the Bloch sphere, and Bob’s goal is to produce a state \( |\hat{n}'\rangle \) that maximizes the average fidelity \( \langle |\hat{n}| |\hat{n}'\rangle \rangle \). Under these restrictions show that the maximum average fidelity \( \int d\hat{n} |\langle \hat{n}| \hat{n}' \rangle|^2 \) is 1/2.

(b) Next we consider strategies that allow Alice to classically communicate with Bob, just like in standard teleportation. Suppose that Alice measures her generated state \( |\hat{n}\rangle \) in the \( \{ |0\rangle, |1\rangle \} \) basis and obtains outcome \( a \), for \( a \in \{0, 1\} \). She computes to Bob the outcome of her measurement, and then Bob produces the state \( |\hat{n}'\rangle = |a\rangle \). What is average fidelity \( |\langle \hat{n}| \hat{n}' \rangle|^2 \) obtained using such a strategy?

(Bibliography note. The problem of simulating teleportation with no entanglement was considered by Popescu in Ref. [Pop94], where he presented the protocol described in part (b). Optimality of this protocol was later proven by Massar and Popescu in Ref. [MP95].)
References

