(1) Suppose that Alice and Bob share the state $|\psi\rangle = \sqrt{\frac{1}{3}} (|00\rangle + |01\rangle + |10\rangle)$. The agree to perform the following two-round LOCC protocol.

1. Alice measures in the basis $\{|0\rangle, |1\rangle\}$ and tells Bob her result.

2. If Alice obtains outcome $|0\rangle$, then Bob measures in the $\{|0\rangle, |1\rangle\}$ basis. If she obtains outcome $|1\rangle$, then he measures in the $\{|+, -\rangle\}$ basis.

Write out the density matrix for the final state averaged over all outcomes (i.e. assume that Alice and Bob discard all measurement outcomes). Is the state a product state? Is the state entangled?

**SOLUTION:**

There are four possible outcomes which amounts to performing the four projectors $\{|00\rangle\langle 00|_{AB}, |01\rangle\langle 01|_{AB}, |1+\rangle\langle 1+|_{AB}, |1-\rangle\langle 1-|_{AB}\}$. By applying the projectors to $|\psi\rangle$ respectively, we find that the four (unnormalized) post-measurement states $\{\sqrt{\frac{1}{3}}|00\rangle, \sqrt{\frac{1}{3}}|01\rangle, \sqrt{\frac{1}{6}}|1+\rangle, -\sqrt{\frac{1}{6}}|1-\rangle\}$. Hence the average final state is

$$\rho_{AB} = \frac{1}{3} |00\rangle\langle 00| + \frac{1}{3} |01\rangle\langle 01| + \frac{1}{6} |1+\rangle\langle 1+| + \frac{1}{6} |1-\rangle\langle 1-|$$

$$= \frac{1}{3} |0\rangle\langle 0| \otimes I/2 + \frac{1}{6} |1\rangle\langle 1| \otimes I/2$$

$$= \left( \frac{1}{3} |0\rangle\langle 0| + \frac{1}{6} |1\rangle\langle 1| \right) \otimes I/2. \quad (1)$$

This is a product state.

(2) Let $|\Phi^+_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$. For an arbitrary unitary $U$ acting on $C^d$, prove that $U \otimes U^* |\Phi^+_d\rangle = |\Phi^+_d\rangle$,

where $U^*$ is the complex conjugate of $U$ in the computational basis.
SOLUTION:

Express $U$ in the computational basis as $U = \sum_{j,k=1}^d u_{jk} |j\rangle \langle k|$ so that $U|\psi\rangle = \sum_{j=1}^d u_{ji} |j\rangle$. Note that the transpose of $U$ is $U^T = \sum_{j,k=1}^d u_{jk} |j\rangle \langle k|$ and $U^T |\psi\rangle = \sum_{j=1}^d u_{jk} |j\rangle$. Then

$$U \otimes I |\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d U|i\rangle \otimes |i\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d \sum_{j=1}^d u_{ji} |j\rangle \otimes |i\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^d \sum_{j=1}^d |j\rangle \otimes u_{jk} |k\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^d |j\rangle \otimes U^T |j\rangle = I \otimes U^T |\Phi_d^+\rangle. \quad (2)$$

Applying $U^*$ on the second system to both sides of this equality gives $U \otimes U^* |\Phi_d^+\rangle = |\Phi_d^+\rangle$.

(3)

Recall the $U \otimes U^*$-twirling

$$\mathcal{T}_{U \otimes U^*}(\rho) = \frac{1}{|\mathcal{G}|} \sum_{U_i \in \mathcal{G}} (U_i \otimes U_i^*) \rho (U_i \otimes U_i^*)^+$$

that transforms an arbitrary $\rho^{AB}$ into a $U \otimes U^*$-invariant state:

$$\rho \rightarrow \rho_\lambda := \lambda \Phi_d^+ + \frac{1-\lambda}{d^2} I \otimes I \quad 0 \leq \lambda \leq 1.$$ 

(a) Every $U \otimes U^*$ can also be expressed as

$$\rho_f := f \Phi_d^+ + \frac{1-f}{d^2-1} (I \otimes I - \Phi_d^+)$$

What is the relationship between the parameters $\lambda$ and $f$ when $\rho_\lambda = \rho_f$?

SOLUTION:

(a) Expand $I \otimes I = (I \otimes I - \Phi_d^+) + \Phi_d^+$ so that we can write

$$\rho_\lambda = (\lambda + \frac{1-\lambda}{d^2}) \Phi_d^+ + \frac{1-\lambda}{d^2} I \otimes I.$$ 

Since $\Phi_d^+$ and $(I \otimes I - \Phi_d^+)$ are orthogonal, we can compare the coefficients of $\rho_\lambda$ and $\rho_f$ to see the relationship

$$f = \lambda + \frac{1-\lambda}{d^2}. \quad (3)$$
(b) $U \otimes U^*$-twirling is unable to generate entanglement. But does it preserve entanglement? In other words, if $\rho^{AB}$ is entangled, will $\mathcal{T}_{U \otimes U^*}(\rho)$ also be entangled?

**SOLUTION:**

The action of twirling transforms a state $\rho$ as

$$\rho \rightarrow \mathcal{T}_{U \otimes U^*}(\rho) = f\Phi_d^+ + \frac{1-f}{d^2-1}(I \otimes I - \Phi_d^+),$$

where $f = \langle \Phi_d^+ | \rho | \Phi_d^+ \rangle$. We know that the resulting state is not entangled when $f \leq 1/d$. So let us choose $d = 2$ and consider the entangled state $|\Phi^\perp\rangle$. Since $f = |\langle \Phi^\perp | \Phi^+ \rangle|^2 = 0$, we have that twirling $|\Phi^\perp\rangle$ will turn it into a separable state.

(c) For every value of $0 \leq \lambda \leq 1$, find a bipartite quantum state $\sigma_\lambda$ such that $\mathcal{T}_{U \otimes U^*}(\sigma_\lambda) = \rho_\lambda$.

**SOLUTION:**

This is trivial since isotropic state is invariant under twirling. So for any $\lambda$, choose the state $\rho_\lambda$ and it will satisfy $\mathcal{T}_{U \otimes U^*}(\rho_\lambda) = \rho_\lambda$.

4)

Consider the bipartite Hilbert space $C^d \otimes C^d$. The **symmetric subspace** $S_+$ of $C^d \otimes C^d$ consists of all states $|\psi\rangle$ such that $F|\psi\rangle = |\psi\rangle$, where $F = \sum_{i,j=1}^d |ij\rangle\langle ji|$ is the swap operator. The **anti-symmetric subspace** $S_-$ of $C^d \otimes C^d$ consists of all states $|\psi\rangle$ such that $F|\psi\rangle = -|\psi\rangle$.

(a) Show that every bipartite state $|\Psi\rangle$ can be expressed as $|\Psi\rangle = \alpha|\varphi_+\rangle + \beta|\varphi_-\rangle$ where $|\varphi_+\rangle \in S_+$ and $|\varphi_-\rangle \in S_-$. 

**SOLUTION:**

We have the identity

$$|\Phi\rangle = \frac{1}{2}(|\Phi\rangle + F|\Phi\rangle) + \frac{1}{2}(|\Phi\rangle - F|\Phi\rangle) = \alpha|\varphi_+\rangle + \beta|\varphi_-\rangle$$

where $|\varphi_+\rangle = \frac{1}{2\alpha}(|\Phi\rangle + F|\Phi\rangle)$ and $|\varphi_-\rangle = \frac{1}{2\beta}(|\Phi\rangle + F|\Phi\rangle)$ for normalization factors $\alpha$ and $\beta$. Note that $|\varphi_+\rangle$ is a symmetric state and $|\varphi_-\rangle$ is an anti-symmetric state.

(b) What are the dimensions of $S_+$ and $S_-$? Find an orthonormal basis for $S_+$ and $S_-$. 

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3
Let us use the two-qubit state space as a guiding example. In $\mathbb{C}^2 \otimes \mathbb{C}^2$, the symmetric subspace is spanned by the vectors \{\ket{00}, \ket{11}, \ket{01} + \ket{10}\} and the anti-symmetric subspace is spanned by the singlet \{\ket{01} - \ket{10}\}. Generalizing now to $\mathbb{C}^d \otimes \mathbb{C}^d$, the symmetric subspace will be spanned by two types of vectors, those belonging to the set \{\ket{ii}\} \bigcup J and those belonging to the set \{\sqrt{1/2}(\ket{ij} + \ket{ji})\} \bigcup I. In total the first set contains $d$ elements and the second set contains $d(d-1)/2$ elements. So the dimension of $S_+$ is $d + d(d-1)/2$. For the anti-symmetric subspace, a basis is given by the vectors \{\sqrt{1/2}(\ket{ij} - \ket{ji})\} \bigcup I. There are a total of $d(d-1)/2$ vectors in this set, so the dimension of $S_-$ is $d(d-1)/2$.

(c) Show that the subspace projectors onto $S_+$ and $S_-$ can be expressed as

\[ P_+ = \frac{1}{2}(I + F), \quad P_- = \frac{1}{2}(I - F). \]  

respectively.

**SOLUTION:**

For part (b), we can directly compute

\[ P_+ = \sum_{i=1}^{d} \ket{ii}\bra{ii} + \frac{1}{2} \sum_{i>j} (\ket{ij} + \ket{ji})(\bra{ij} + \bra{ji}) \]

\[ = \frac{1}{2} \sum_{i=1}^{d} \ket{ii}\bra{ii} + \frac{1}{2} \sum_{i>j} (\ket{ij}\bra{ij} + \ket{ji}\bra{ji}) \]

\[ = \frac{1}{2} (I + F). \]  

(5)

A similar calculation gives

\[ P_- = \sum_{i=1}^{d} \ket{ii}\bra{ii} + \frac{1}{2} \sum_{i>j} (\ket{ij} - \ket{ji})(\bra{ij} - \bra{ji}) \]

\[ = \frac{1}{2} \sum_{i=1}^{d} \ket{ii}\bra{ii} + \frac{1}{2} \sum_{i>j} (\ket{ij}\bra{ij} - \ket{ji}\bra{ji}) \]

\[ = \frac{1}{2} (I - F). \]  

(6)

**U \otimes U-Invariant States.**

In this problem you will characterize the family of $U \otimes U$-invariant states. For $d \otimes d$ quantum systems, these are the states satisfying

\[ \rho^{AB} = (U \otimes U)\rho^{AB}(U \otimes U)^\dagger \quad \forall d \times d \text{ unitaries } U. \]  

(7)
(a) Show that a general \( U \otimes U \)-invariant state has the form

\[
\omega_\lambda = \frac{2(1 - \lambda)}{d(d + 1)} P_+ + \frac{2\lambda}{d(d - 1)} P_-,
\]

where \( P_+ \) and \( P_- \) are projectors onto the symmetric and anti-symmetric subspaces, respectively (see Eq. (4)).

**SOLUTION:**

Let \( \langle rs|\rho|pq \rangle \) be the elements of \( \rho \) in the computational basis. By considering unitaries of the form \( f_s|t\rangle = (-1)^{\delta_{st}}|t\rangle \), we are immediately able to eliminate all components except those having the form

\[
\langle ss|\rho|pp \rangle, \quad \langle sp|\rho|sp \rangle, \quad \langle sp|\rho|ps \rangle, \quad \langle ss|\rho|ss \rangle.
\]

Invariance under unitaries of the form \( K_s|t\rangle = (i)^{\delta_{st}}|t\rangle \) require that the \( \langle ss|\rho|pp \rangle \) terms vanish. Because of invariance under permutation, the three remaining terms must be constant for each choice of \( s \) and \( p \). Thus we have \( \rho \) down to the form

\[
\rho = a \sum_{s \neq p} |sp\rangle \langle sp| + b \sum_{s \neq p} |sp\rangle \langle ps| + c \sum_{s=1}^d |ss\rangle \langle ss|.
\]

For a fixed \( s \) and \( p \), apply the rotation \( |s\rangle \rightarrow \frac{1}{\sqrt{2}}(|s\rangle + |t\rangle) \) and \( |t\rangle \rightarrow \frac{1}{\sqrt{2}}(|s\rangle - |t\rangle) \) while keeping all other states unchanged. For \( \rho \) to remain invariant under this transformation, we need

\[
a(|sp\rangle \langle sp| + |ps\rangle \langle ps|) + b(|sp\rangle \langle ps| + |ps\rangle \langle sp|) + c(|ss\rangle \langle ss| + |pp\rangle \langle pp|) =
\]

\[
= \frac{1}{4} a ((|ss\rangle - |sp\rangle + |ps\rangle - |pp\rangle)(|ss\rangle - |sp\rangle + |ps\rangle - |pp\rangle))
\]

\[
+ \frac{1}{4} a ((|ss\rangle + |sp\rangle - |ps\rangle - |pp\rangle)(|ss\rangle + |sp\rangle - |ps\rangle - |pp\rangle))
\]

\[
+ \frac{1}{4} b ((|ss\rangle - |sp\rangle + |ps\rangle - |pp\rangle)(|ss\rangle + |sp\rangle - |ps\rangle - |pp\rangle))
\]

\[
+ \frac{1}{4} b ((|ss\rangle + |sp\rangle - |ps\rangle - |pp\rangle)(|ss\rangle - |sp\rangle + |ps\rangle - |pp\rangle))
\]

\[
+ \frac{1}{4} c ((|ss\rangle + |sp\rangle + |ps\rangle + |pp\rangle)(|ss\rangle + |sp\rangle + |ps\rangle + |pp\rangle))
\]

\[
+ \frac{1}{4} c ((|ss\rangle - |sp\rangle - |ps\rangle + |pp\rangle)(|ss\rangle - |sp\rangle - |ps\rangle + |pp\rangle))
\]

Equality holds iff \( c = a + b \). Hence

\[
\rho = a \sum_{s,p=1}^d |sp\rangle \langle sp| + b \sum_{s,p=1}^d |sp\rangle \langle ps|
\]

\[
= aI + bF = a(P_+ + P_-) + b(P_+ - P_-) = (a + b)P_+ + (a - b)P_-. \tag{9}
\]
Trace both sides to obtain the normalization

\[ 1 = (a + b)(d + \binom{d}{2}) + (a - b)\binom{d}{2} = (a + b)\frac{d(d + 1)}{2} + (a - b)\frac{d(d - 1)}{2}. \]

Let \( a + b = (1 - \lambda)\frac{2}{d(d + 1)} \) and \( a - b = \lambda\frac{2}{d(d - 1)} \) for a parameter \( \lambda \). Thus a \( U \otimes U \) invariant state has the form

\[ \omega_\lambda = \frac{2(1-\lambda)}{d(d + 1)} P_+ + \frac{2\lambda}{d(d - 1)} P_- . \]

(b) Provide a range of \( \lambda \) for which \( \omega_\lambda \) is separable.

**SOLUTION:**

To find a range of \( \lambda \) for which \( \omega_\lambda \) is separable, we can twirl the product state \( |00\rangle\langle00| \). Since twirling is an LOCC operations, the resulting state will not be entangled. The symmetric and anti-symmetric projectors are obviously \( U \otimes U \) invariant, so we have that

\[ 1 = \text{Tr}[P^+|00\rangle\langle00|] = \text{Tr}[P^+ T_{U \otimes U}(|00\rangle\langle00|)] = (1 - \lambda). \quad (10) \]

Hence \( \omega_\lambda \) must be separable when \( \lambda = 0 \). Now, let us try twirling the state \( |01\rangle\langle01| \). By the same reasoning, we have

\[ 1/2 = \text{Tr}[P^+|01\rangle\langle01|] = \text{Tr}[P^+ T_{U \otimes U}(|01\rangle\langle01|)] = (1 - \lambda). \quad (11) \]

This shows that \( \omega_\lambda \) is separable when \( \lambda = 1/2 \). Thus by considering the action of twirling any convex combination of \( |00\rangle\langle00| \) and \( |11\rangle\langle11| \), we find that \( \omega_\lambda \) is separable in the interval \( 0 \leq \lambda \leq 1/2 \).

(c) For two-qubits, show that \( U \otimes U^* \) and \( U \otimes U \)-invariant states are related by a local unitary.

**SOLUTION:**

For two-qubits, the \( U \otimes U^* \)-invariant states have the form \( \lambda \Phi^+ + (1 - \lambda)/3(\mathbb{1} \otimes \mathbb{1} - \Phi^+) \), while the \( U \otimes U \)-invariant states have the form \( \lambda \Psi^- + (1 - \lambda)(\mathbb{1} - \Psi^-) \). The states are related by the local unitary \( \sigma_y \otimes I \), which transforms \( |\Phi^+\rangle \) into \( |\Psi^-\rangle \).