Homework 4

(1)
Suppose that Alice and Bob share the state $|\psi\rangle = \sqrt{\frac{1}{3}}(|00\rangle + |01\rangle + |10\rangle)$. The agree to perform the following two-round LOCC protocol.

1. Alice measures in the basis $\{|0\rangle, |1\rangle\}$ and tells Bob her result.

2. If Alice obtains outcome $|0\rangle$, then Bob measures in the $\{|0\rangle, |1\rangle\}$ basis. If she obtains outcome $|1\rangle$, then he measures in the $\{|+\rangle, |--\rangle\}$ basis.

Write out the density matrix for the final state averaged over all outcomes (i.e. assume that Alice and Bob discard all measurement outcomes). Is the state a product state? Is the state entangled?

(2)
Let $|\Phi^+_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$. For an arbitrary unitary $U$ acting on $\mathbb{C}^d$, prove that

$$U \otimes U^* |\Phi^+_d\rangle = |\Phi^+_d\rangle,$$

where $U^*$ is the complex conjugate of $U$ in the computational basis.

(3)
Recall the $U \otimes U^*$-twirling

$$T_{U \otimes U^*}(\rho) = T_G(\rho) = \frac{1}{|G|} \sum_{U_i \in G} (U_i \otimes U_i^*) \rho (U_i \otimes U_i^*)^\dagger$$

that transforms an arbitrary $\rho^{AB}$ into a $U \otimes U^*$-invariant state:

$$\rho \rightarrow \rho_\lambda := \lambda \Phi^+_d + \frac{1 - \lambda}{d^2} I \otimes I \quad 0 \leq \lambda \leq 1.$$

(a) Every $U \otimes U^*$ can also be expressed as

$$\rho_f := f \Phi^+_d + \frac{1 - f}{d^2 - 1} (I \otimes I - \Phi^+_d).$$

What is the relationship between the parameters $\lambda$ and $f$ when $\rho_\lambda = \rho_f$?

(b) $U \otimes U^*$-twirling is unable to generate entanglement. But does it preserve entanglement? In other words, if $\rho^{AB}$ is entangled, will $T_{U \otimes U^*}(\rho)$ also be entangled?

(c) For every value of $0 \leq \lambda \leq 1$, find a bipartite quantum state $\sigma_\lambda$ such that $T_{U \otimes U^*}(\sigma_\lambda) = \rho_\lambda$. 

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Consider the bipartite Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$. The symmetric subspace $S_+$ of $\mathbb{C}^d \otimes \mathbb{C}^d$ consists of all states $|\psi\rangle$ such that $F|\psi\rangle = |\psi\rangle$, where $F = \sum_{i,j=1}^d |ij\rangle\langle ji|$ is the swap operator. The anti-symmetric subspace $S_-$ of $\mathbb{C}^d \otimes \mathbb{C}^d$ consists of all states $|\psi\rangle$ such that $F|\psi\rangle = -|\psi\rangle$.

(a) Show that every bipartite state $|\Psi\rangle$ can be expressed as $|\Psi\rangle = \alpha |\varphi_+\rangle + \beta |\varphi_-\rangle$ where $|\varphi_+\rangle \in S_+$ and $|\varphi_-\rangle \in S_-$. 

(b) What are the dimensions of $S_+$ and $S_-$? Find an orthonormal basis for $S_+$ and $S_-$. 

(c) Show that the subspace projectors onto $S_+$ and $S_-$ can be expressed as

$$P_+ = \frac{1}{2} (I + F), \quad P_- = \frac{1}{2} (I - F). \quad (1)$$

respectively.

(Hint. In two qubits, the symmetric subspace is spanned by $\{|00\rangle, |11\rangle, \sqrt{1/2} (|01\rangle + |10\rangle)\}$ while the antisymmetric subspace is one-dimensional and spanned by $\sqrt{1/2} (|01\rangle - |10\rangle)$. Generalize this structure for $d$ dimensions.)

(U $\otimes$ U-Invariant States.

In this problem you will characterize the family of $U \otimes U$-invariant states. For $d \otimes d$ quantum systems, these are the states satisfying

$$\rho^{AB} = (U \otimes U) \rho^{AB} (U \otimes U)^\dagger \quad \forall d \times d \text{ unitaries } U. \quad (2)$$

(a) Show that a general $U \otimes U$-invariant state has the form

$$\omega_\lambda = \frac{2(1 - \lambda)}{d(d + 1)} P_+ + \frac{2\lambda}{d(d - 1)} P_- \quad (3)$$

where $P_+$ and $P_-$ are projectors onto the symmetric and anti-symmetric subspaces, respectively (see Eq. (1)).

(b) Provide a range of $\lambda$ for which $\omega_\lambda$ is separable.

(c) For two-qubits, show that $U \otimes U^*$ and $U \otimes U$-invariant states are related by a local unitary.

(Hint. Carefully follow the steps derived in class for $U \otimes U^*$-invariant states.)

(Bibliography note. The class of $U \otimes U$-invariant states were first studied by R. Werner in Ref. [Wer89], and they are commonly referred to as Werner states. The $U \otimes U^*$-invariant isotropic states were later introduced by the Horodeckis in Ref. [HH99].)
References
