Homework 5

(1) **Fun with convexity**

A single-variable function $f : \mathbb{R} \to \mathbb{R}$ is called convex if

$$
\sum_i p_i f(x_i) \geq f(\sum_i p_i x_i)
$$

where $(p_i)$ is a probability distribution and the $x_i \in \mathbb{R}$ are arbitrary.

(a) Suppose that $f(x)$ is a convex function over some range $[a, b]$. Show that every point on the line connecting $f(a)$ and $f(b)$ is greater than $f(c)$ for any $c \in (a, b)$.

(b)∗ Prove that the two-qubit entanglement entropy $E(C) = h \left( \frac{1}{2} \left( 1 + \sqrt{1 - C^2} \right) \right)$ is a monotonically increasing, convex function of the concurrence $C$.

(2) Prove that the concurrence $C(\Psi)$ is a pure state entanglement monotone.

(*Hint. Recall that a function is a pure state monotone if it is a concave function of the Schmidt coefficients.)*

(3) Compute the entanglement of formation for all two-qubit Isotropic and Werner states; i.e. the $U \otimes U^*$-invariant and $U \otimes U$-invariant two-qubit states.

(*Hint. Apply our calculation for the entanglement of formation for Bell-diagonal states that we derived in class.)*

(4) Show that $F^+_d(\rho) \leq \frac{1}{3}$ for any separable state $\rho$, where $F^+_d(\rho)$ is the maximally entangled fraction.

(*Hint. Use the fact that $F^+_d(\rho)$ is a linear function of $\rho$. If $\rho$ is separable, it can be written as a convex combination of pure product states. Hence by linearity, an upper bound on $F^+_d(\rho)$ can be obtained by computing $F^+_d(|\alpha\rangle\langle\alpha| \otimes |\beta\rangle\langle\beta|)$ for an arbitrary product state $|\alpha\rangle|\beta\rangle$.*

References