An Appetizer: Hardy’s Paradox

Suppose that Alice and Bob each have a “black box.” Each box has a button that, when pressed, causes either a red or green bulb to flash on the box. The contents of these boxes and how they work is unknown to Alice and Bob, but they can still infer some general behavior of their boxes by observing what colors flash when they press their respective buttons.

Alice begins running “experiments” on her box by pressing her button and recording the resulting color. After 100 such experiments, she finds that the red bulb flashed 49 times while the green bulb flashed 51 times. Based on these outcomes, she hypothesizes that her box behaves like an unbiased coin flip that flashes either red or green with an equal probability of 1/2. Each press of the button generates a completely random color flash, and if she were to continue this experiment for 100 more iterations, then the fraction of red to green flashes should approach 1 even closer.

It turns out that Bob was also experimenting with his box, and he found the red and green bulbs to likewise flash roughly an equal number of times. However, when Alice and Bob compare their sequences of outcomes, they find that their boxes flashed the exact same colors at each iteration of the experiment. Surprised by this discovery, they continue with more experiments. Whenever Alice and Bob press their buttons within a second of each other, they obtain the exact same color.

Bob finds this behavior astonishing. He exclaims:

Alice, we have boxes that are able to instantaneously communicate with each other. On its own, my box flashes red 50% of the time and green the other 50%. However, when your box flashes, somehow my box immediately knows what color it is, and it does the exact same. Imagine we were located on planets light-years apart. Whenever you press your button, I could then instantaneously learn your outcome by pressing the button on my box. I’ve heard of this before. It’s the magic of quantum mechanics, and this is called ‘spooky action at a distance.’

Alice pauses for a moment, thinks it over, and then responds.

Bob, I don’t think this is ‘spooky action at a distance,’ and I don’t see at all what this has to do with quantum mechanics. True, your box flashes random colors on its own, but my flashes are random as well. At most we can conclude that our boxes are perfectly coordinated to flash the same random color. Here’s one explanation for how this could happen. Suppose our boxes came from the same factory, and when they were constructed, the manufacturers generated a very, very long string of random R’s and G’s. Inside each of our boxes is an internal clock, and the string of R’s and G’s functions as instructions that is coordinated with the clock. When the button is pressed \( n \) seconds after the internal clock started, the box flashes the \( n^{th} \) letter in the instructional string, \( R \) being red and \( G \) being green. Now if both our boxes were given the same instructions and our internal clocks started at the same time, then we would always flash the same color. Furthermore, presuming the instruction string has 50% \( R \)’s and 50% \( G \)’s, then our color flashes would also appear to be random.
Alice is quite right in her reply to Bob. The joint behavior of Alice and Bob’s boxes does not exemplify “spooky action at a distance,” although many popular science books and articles will attempt to describe the latter in such a way. The phrase “spooky actions at a distance” was used by Einstein in a letter to Max Born for the purpose of characterizing his criticism with quantum mechanics [EBB71]. The source of Einstein’s skepticism lies in the phenomenon now referred to as quantum nonlocality. Quantum nonlocality is a consequence of the ability for quantum systems to exist in states that are completely unfamiliar to our everyday experience of classical physics. Quantum systems in such states are said to be entangled, and the theory of quantum entanglement is the focus of this course. We will see in Chapter IX what it means precisely for quantum systems to demonstrate nonlocality, and how this relates to entanglement.

For the time being, let us try to appreciate some of Einstein’s discomfort by examining how quantum entanglement can lead to counterintuitive physical behavior. We will do this by modifying the “black box” experiment described above. Suppose that Alice and Bobs’ boxes each have two “input settings” labeled 0 and 1. For each setting, the boxes still flash a color whenever the button is pressed. Alice and Bob can again run their experiments, but now there are more variables. Under each iteration of the experiment, they must record their respective input setting (0/1) as well as their output color flash (R/G). Let us suppose that after 1,000 trials of Alice and Bob independently choosing random 0/1 inputs, their “black box” statistics resemble the following chart:

<table>
<thead>
<tr>
<th>Input \ Output</th>
<th>(R, R)</th>
<th>(R, G)</th>
<th>(G, R)</th>
<th>(G, G)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>0</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>2/3</td>
<td>0</td>
<td>1/6</td>
<td>1/6</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>2/3</td>
<td>1/6</td>
<td>0</td>
<td>1/6</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>3/4</td>
<td>1/12</td>
<td>1/12</td>
<td>1/12</td>
</tr>
</tbody>
</table>

Table 1: Alice and Bob’s proportions of outcomes after 1,000 trials using randomly chosen inputs.

In this chart, row (1, 0) and column (R, G), for example, conveys the information that when Alice chooses setting 1 and Bob chooses setting 0, a fraction 1/6 of the time Alice obtains a red flash while Bob obtains a green flash. The other columns in that row quantify the frequency of the other three outcomes given the same input settings.

Having closely examining the table, Bob again turns to Alice and says:

Okay, Alice, I agree with your previous explanation. But I think there really is something strange about this new experimental data. Hear me out:

(a) The second row says that it is possible for me to obtain output G when I choose input 1, and when I do, you could not have obtained R if you had chose input 0.

(b) Likewise, the third row says that it is possible for you to obtain output G when you choose input 1, and when you do, I could not have obtained R if I had chose input 0.

(c) But now the fourth row says that it is possible for us to both obtain output G when choosing input 1, and when we do, (a) and (b) imply that we could not have both obtained R had we both chosen input 0 instead.

(d) But this directly contradicts the first row which says that whenever we both choose input 0, we both obtain R a fraction 1/3 of the time.

Therefore, I conclude this data is physically inconsistent.
Alice pauses even longer this time and carefully scrutinizes Bob’s argument. She replies:

Bob, you are assuming a certain property of our boxes that intuitively seems justified, but it might not be true. In steps (a) and (b) of your argument, you are making counterfactual claims about how our boxes would have behaved had things been otherwise. For this to be necessarily true, our boxes have to have a causal structure at each iteration of the experiment dictating what color will flash for each of our possible input settings. In other words, going back to my earlier explanation, you are assuming that whenever we perform the experiment and prior to our choice of input settings, there is some instruction set that stipulates how our bulbs will flash for all four possible combinations of input choices.

But quantum mechanics tells us that such prior specification of potential outcomes is not always possible. Specifically, there are certain pairs of quantum properties that cannot both be determined prior to measurement; this is the famous Heisenberg Uncertainty Principle. When measuring one of these properties, it is impossible to say what outcome would have been obtained if we had instead measured the other.

So if our “black boxes” are quantum systems and our flashing lights represent the outcome of quantum measurement, then your counterfactual claims in (a) and (b) are not true. All we can conclude from, say, the second row is that when we both input 0, our boxes never both flash G.

Once again, Alice’s wisdom and basic understanding of quantum mechanics shines through. If we assume, like Bob, that all measurable properties of a physical system are specified at each moment in time, then the measurement data in Table 1 is impossible. Such an assumption lies at the foundation of classical Newtonian physics where the physical properties of a closed system are entirely determined by the position and momenta of its constituent parts at each instance in time. Hence measurement data resembling Table 1 can only be generated by measuring quantum systems. In particular, the measured systems must be entangled.

The argument described here was first presented by Lucien Hardy in 1993 [Har93], and it is often referred to as “Hardy’s Paradox.” A simple entangled state capable of generating the statistics of Table 1 is given by

$$|\psi_{\text{Hardy}}\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle).$$

(1)

In this course you will learn why $|\psi_{\text{Hardy}}\rangle$ represents an entangled quantum state and how it can lead to the measurement data of Table 1.

References
