LOCC and Entanglement

PHYS 500 - Southern Illinois University

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What is Quantum Entanglement?

Quantum entanglement will be defined in terms of LOCC.

Recall, that LOCC consists of measurements having product Kraus operators \( \{ A_\lambda \otimes B_\lambda \}_\lambda \) built from two types of basic quantum operations:

1. Local quantum measurements
2. Classical communication of measurement results

Included in LOCC is the “discarding” or “forgetting” of measurement data.

If Alice makes a local measurement \( \{ M_k \}_k \) on her system and then discards her result, we take the ensemble average:

\[
\rho_{AB} \xrightarrow{\text{Measurement}} \rho_k^{AB} = (M_k^A \otimes I^B) \rho_{AB} (M_k^A \otimes I^B)^\dagger / p_k
\]

\[
\rho_{AB} \xrightarrow{\text{Discarding}} \sum_k p_k \rho_k^{AB} = \sum_k (M_k^A \otimes I^B) \rho_{AB} (M_k^A \otimes I^B)^\dagger (1)
\]
What is Quantum Entanglement?

An operational definition of entanglement:
Entanglement is what cannot be generated by LOCC.

More precisely:

* A state $\rho^{AB}$ is said to be **entangled** if there does not exist an LOCC protocol that transforms a product state into $\rho^{AB}$.

Alice can freely generate any state $\sigma^A$ in her lab, and likewise Bob can freely generate any state $\sigma^B$ in his lab.

If there exists an LOCC protocol $\mathcal{L}$ such that $\sigma^A \otimes \sigma^B \xrightarrow{\mathcal{L}} \rho^{AB}$, then $\rho^{AB}$ is **not** entangled.
Separable States

**Theorem**

A state $\rho^{AB}$ is not entangled if and only if it can be expressed as a *convex combination* of product states

$$\rho^{AB} = \sum_i p_i \sigma_i^A \otimes \sigma_i^B.$$ 

In general, for a vector space $V$ with vectors $v_i \in V$, a *convex combination* is any linear combination of the form $\sum_{i=1}^n p_i v_i$, where $p_i > 0$ and $\sum_{i=1}^n p_i = 1$. These are also called “probabilistic mixtures.”

Every density matrix is a convex combination of orthogonal projectors:

$$\sigma_i^A = \sum_{j=1}^{r_i} q_{i,j} |a_{i,j}\rangle \langle a_{i,j}|,$$

$$\sigma_i^B = \sum_{k=1}^{s_i} q_{i,k} |b_{i,k}\rangle \langle b_{i,k}|.$$
Separable States

Theorem

A state $\rho^{AB}$ is not entangled if and only if it can be expressed as a convex combination of product states

$$
\rho^{AB} = \sum_i p_i \sum_{j=1}^{r_i} q_{i,j} |a_{i,j}\rangle \langle a_{i,j}| \otimes \sum_k q_{i,k} |b_{i,k}\rangle \langle b_{i,k}|
$$

$$
= \sum_{i,j,k} p_i q_{i,j} q_{i,k} |a_{i,j}\rangle \langle a_{i,j}| \otimes |b_{i,k}\rangle \langle b_{i,k}|
$$

$$
= \sum_\lambda p_\lambda |\alpha_\lambda\rangle \langle \alpha_\lambda| \otimes |\beta_\lambda\rangle \langle \beta_\lambda|.
$$

States of this form are called a separable states.

The $\{|\alpha_\lambda\rangle\}_\lambda$ are not necessarily orthogonal, and likewise for the $\{|\beta_\lambda\rangle\}_\lambda$. 
Separable States

Separable states are convex combinations of pure product states. The set of separable states is the set of unentangled states. **Diagram:**

To prove the above theorem, we must show that

1. Every LOCC protocol performed on a product state generates a separable state,
2. Conversely, every separable state can be generated from some product state using LOCC.
Separable States

Proof:
Separable States

For a pure state $|\Psi\rangle^{AB} = \sum_{i,j} c_{ij} |i\rangle^A |j\rangle^B$, it is relatively easy to determine whether it’s entangled:

We form the matrix $\sum_{i,j} c_{i,j} |i\rangle \langle j|$ and compute its rank. The state is entangled iff the rank is greater than one.

For a mixed state $\rho^{AB}$, it is much more difficult (NP-Hard) to determine whether it’s entangled:

We must show that $\rho^{AB}$ cannot be expressed as a convex combination of pure product states.

The difficulty arises because there are an infinite number of pure state decompositions for every mixed state. For entanglement, at least one entangled state must belong to every pure state decomposition.
Example
Let

\[ |\psi_1\rangle = \frac{1}{\sqrt{3}} \left( \frac{3}{2} |00\rangle + \frac{1}{2} (|01\rangle + |10\rangle + |11\rangle) \right) \]

\[ |\psi_2\rangle = \frac{1}{\sqrt{3}} \left( \frac{3}{2} |00\rangle - \frac{1}{2} (|01\rangle + |10\rangle + |11\rangle) \right). \]

Is the state \( \rho^{AB} = \frac{1}{2} (|\psi_1\rangle\langle \psi_1 | + |\psi_2\rangle\langle \psi_2 |) \) entangled?
Quantifying Entanglement

The next objective will be to quantify the amount of entanglement in a given state. We want to be able to say *how much* entanglement a state possesses.

Again we will adopt an operational philosophy and measure entanglement in terms of LOCC.

To begin, let us focus on two extremes: No Entanglement and Maximal Entanglement.

As discussed above, every separable state possesses zero entanglement.

What about for maximal entanglement?
Maximal Entanglement

We will say that a $d \otimes d$-dimensional bipartite state $\rho^{AB}$ has maximal entanglement if it can be transformed into any other $d \otimes d$-dimensional state $\sigma^{AB}$ using LOCC:

$$\rho^{AB} \xrightarrow{\text{LOCC}} \sigma^{AB}.$$ 

Theorem

For a $d \otimes d$ system, the state $|\Phi^+_d\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |kk\rangle^{AB}$ has maximal entanglement, as well as any other state related to $|\Phi^+_d\rangle$ by a local unitary (LU) transformation.

Proof

We prove just for two qubits. The more general statement will be proven later.