Linear Algebra and Dirac Notation, Pt. 1

February 1, 2017
Motivation

In this lecture we develop the basic tools to study the mathematical theory of finite-dimensional quantum mechanics. We will be primarily focused on how this theory can be applied to quantum information processing.
Quantum mechanics describes physical systems in terms of **Hilbert Spaces**. When we refer to Hilbert space in this course, we mean some $d$-dimensional complex vector space $\mathcal{H}$ with a defined **inner product**.

**Definition (Inner Product)**

A function $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is an inner product on vector space $\mathcal{H}$ if for all $v, w \in \mathcal{H}$ and $\lambda_i \in \mathbb{C}$:

1. **Linearity in the second argument:**
   
   \[ \langle v, \sum \lambda_i w_i \rangle = \sum \lambda_i \langle v, w_i \rangle; \]

2. **Conjugate-commutativity:**

   \[ \langle v, w \rangle = \langle w, v \rangle^*; \]

3. **Non-negativity:**

   \[ \langle v, v \rangle \geq 0 \text{ with equality iff } v = 0. \]

Two vectors $v, w \in \mathcal{H}$ are **orthogonal** if $\langle v, w \rangle = 0$.

A vector $v$ is **normalized** if $||v|| = \sqrt{\langle v, v \rangle} = 1$. 
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1. Linearity in the second argument: \( \langle \mathbf{v}, \sum_i \lambda_i \mathbf{w}_i \rangle = \sum_i \lambda_i \langle \mathbf{v}, \mathbf{w}_i \rangle \);
2. Conjugate-commutativity: \( \langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle^* \);
3. Non-negativity: \( \langle \mathbf{v}, \mathbf{v} \rangle \geq 0 \) with equality iff \( \mathbf{v} = 0 \).
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Notation

We will be primarily interested in normalized vectors. We write $\tilde{\psi}$ for an arbitrary $|\tilde{\psi}\rangle \in \mathcal{H}$ and $\psi$ (without the “tilde”) if $|\psi\rangle$ is normalized and parallel to $|\tilde{\psi}\rangle$:

$$|\psi\rangle = \frac{1}{|||\tilde{\psi}\rangle||} |\tilde{\psi}\rangle.$$
For a $d$-dimensional Hilbert space $\mathcal{H}$, we can always construct an orthonormal basis $\{ |b_1\rangle, |b_2\rangle, \ldots |b_d\rangle \}$ with $|b_i\rangle \in \mathcal{H}$ and $\langle b_i|b_j\rangle = \delta_{ij}$. Every element $|\psi\rangle \in \mathcal{H}$ can be written as a linear combination of basis vectors:

$$|\psi\rangle = \sum_{i=1}^{d} x_i |b_i\rangle.$$ 

Normalization requires $1 = \langle \psi|\psi\rangle = \sum_{i=1}^{d} |x_i|^2$. Every vector space has an infinite number of orthonormal bases. In practice, one identifies some particular basis that is easier to work with and calls it the computational basis. Computational basis vectors are denoted by $|0\rangle$, $|1\rangle$, $\ldots$, $|d-1\rangle$. 
Orthonormal Bases

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Proposition

The inner product between any two vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ is given by

$$\langle \psi | \phi \rangle = \sum_{i=1}^{d} y_i^* x_i,$$

where $|\phi\rangle = \sum_{i=1}^{d} x_i |b_i\rangle$, $|\psi\rangle = \sum_{i=1}^{d} y_i |b_i\rangle$, and \{|$b_1\rangle$, $|b_2\rangle$, $\cdots$, $|b_d\rangle$\} is any orthonormal basis for $\mathcal{H}$. 
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**Proposition**

The inner product between any two vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ is given by

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**Proof**
Dual Space and Bras

Definition (Dual Space)
For a Hilbert space $\mathcal{H}$, its dual space $\mathcal{H}^*$ is the vector space consisting of all linear functions acting on $\mathcal{H}$. For every $f \in \mathcal{H}^*$, there exists a unique $|\eta\rangle \in \mathcal{H}$ such that $f(|\psi\rangle) = \langle \eta | \psi \rangle \forall |\psi\rangle \in \mathcal{H}$. The function $f$ is called the dual vector of $|\eta\rangle$ and it is sometimes denoted by the bra $\langle \eta |$. If $\{|b_1\rangle, |b_2\rangle, \ldots, |b_d\rangle\}$ is an arbitrary orthonormal basis and $|\psi\rangle = \sum_{i=1}^{d} x_i |b_i\rangle$, then the bra of $|\psi\rangle$ is given by $\langle \psi | = \sum_{i=1}^{d} x_i^* \langle b_i |$. 
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Remark

Think of the inner product between two vectors $|\psi\rangle$ and $|\phi\rangle$ as “multiplication of a bra onto a ket.”
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For $|\phi\rangle = \sum_{i=1}^{d} x_{i} |b_{i}\rangle$ and $|\psi\rangle = \sum_{i=1}^{d} y_{i} |b_{i}\rangle$, their inner product is

$$\langle\psi|\phi\rangle = \left( \sum_{i=1}^{d} y_{i}^* \langle b_{i}| \right) \left( \sum_{j=1}^{d} x_{j} |b_{j}\rangle \right)$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} y_{i}^* x_{j} \langle b_{i}|(|b_{j}\rangle) = \sum_{i=1}^{d} \sum_{j=1}^{d} y_{i}^* x_{j} \langle b_{i}|b_{j}\rangle$$

$$= \sum_{i=1}^{d} y_{i}^* x_{i}.$$
Dual Space and Bras

In the basis $\{|b_i\rangle\}$, kets are represented by column vectors and bras are represented by row vectors as:

$$|\phi\rangle = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \quad \quad \quad \langle \psi | = \begin{pmatrix} y_1^* & y_2^* & \cdots & y_d^* \end{pmatrix}.$$
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|\phi\rangle = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \quad \quad \langle \psi | = (y_1^* \ y_2^* \ \cdots \ y_d^*)
\]

Their inner product is given by standard matrix multiplication:

\[
\langle \psi | \phi \rangle = (y_1^* \ y_2^* \ \cdots \ y_d^*) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} = \sum_{i=1}^{d} y_i^* x_i.
\]
The Outer Product and Linear Operators

Definition (Linear Operator)

A **linear operator** on $\mathcal{H}$ is a linear transformation $T : \mathcal{H} \rightarrow \mathcal{H}$.
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Definition (Outer Product)
For two vectors $|\psi\rangle$ and $|\phi\rangle$ their **outer product** is the linear operator denoted by $|\psi\rangle\langle\phi|$ and has the action

$$|\psi\rangle\langle\phi| : |\eta\rangle \mapsto |\psi\rangle\langle\phi| |\eta\rangle = \langle\phi|\eta\rangle |\psi\rangle.$$
The Outer Product and Linear Operators

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A **linear operator** on \( \mathcal{H} \) is a linear transformation \( T : \mathcal{H} \rightarrow \mathcal{H} \).

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Remark

Think of the outer product between two vectors \(|\psi\rangle\) and \(|\phi\rangle\) as “multiplication of a ket onto a bra.”
The Outer Product and Linear Operators

An arbitrary linear operator on $\mathcal{H}$ can be written as

$$T = \sum_{i,j} t_{ij} |b_i\rangle\langle b_j|,$$

where $\{|b_1\rangle, |b_2\rangle, \cdots, |b_d\rangle\}$ is an arbitrary orthonormal basis. The numbers $t_{ij}$ are called the components of $T$ with respect to the basis $\{b_i\}$. The matrix representation of $T$ in this basis is given by matrix $T_b$:

$$T \equiv T_b = \begin{pmatrix} t_{11} & t_{12} & \cdots & t_{1d} \\ t_{21} & t_{22} & \cdots & t_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ t_{d1} & t_{d2} & \cdots & t_{dd} \end{pmatrix}, \quad \text{where } t_{ij} = \langle b_i | T | b_j \rangle.$$
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\end{pmatrix}, \quad \text{where } t_{ij} = \langle b_i | T | b_j \rangle.$$

The components are also denoted by $[[T_b]]_{ij} := t_{ij} = \langle b_i | T | b_j \rangle$. 
The Outer Product and Linear Operators

Example

Consider the operator \( T = \frac{1}{4} |0\rangle\langle 0| + \frac{3}{4} |1\rangle\langle 1| \). What is the matrix representation of \( T \) in

(a) The computation basis \( \{ |0\rangle, |1\rangle \} \),
(b) The Hadamard basis \( \{ |+\rangle, |-\rangle \} \), where \( |+\rangle = \sqrt{1/2}( |0\rangle + |1\rangle ) \) and \( |-\rangle = \sqrt{1/2}( |0\rangle - |1\rangle ) \)?
(c) What is the action of \( T \) on the vector \( |\psi\rangle = \cos \theta |+\rangle + \sin \theta |-\rangle \)?
The Adjoint Operator

Definition (Adjoint)

For a linear operator $T$ on $\mathcal{H}$, the **adjoint** of $T$, denoted by $T^\dagger$, is the unique operator satisfying

$$\langle \psi | T^\dagger | \phi \rangle^* = \langle \phi | T | \psi \rangle \quad \forall |\psi\rangle, |\phi\rangle \in \mathcal{H}.$$
Definition (Adjoint)

For a linear operator $T$ on $\mathcal{H}$, the adjoint of $T$, denoted by $T^\dagger$, is the unique operator satisfying

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For an orthonormal basis $\{|b_i\rangle\}_{i=1}^d$, the matrix components of $T$ are related to the matrix components of $T^\dagger$ by

$$[[T_{b}]]_{ij} = \langle b_i | T | b_j \rangle = \langle b_j | T^\dagger | b_i \rangle^* = [[T_{b}^\dagger]]_{ji}^*.$$
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\[
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= \langle b_j | T^\dagger | b_i \rangle^* = [[T^\dagger_{b}]]_{ji}^*.
\]

The matrix representation for \( T^\dagger \) is obtained by taking the conjugate transpose of the matrix representation for \( T \).