Linear Algebra using Dirac Notation: Pt. 2

PHYS 476Q - Southern Illinois University

February 6, 2018
Adjoint Operators

For every operator \( R \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \), there exists a unique operator \( R^\dagger \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1) \) called the **adjoint** of \( R \) that is defined by the condition that

\[
\langle \beta | R^\dagger | \alpha \rangle = \langle \alpha | R | \beta \rangle^* \quad \forall |\alpha\rangle \in \mathcal{H}_1, \quad |\beta\rangle \in \mathcal{H}_2.
\]

Bra-ket rule:

\[
\text{ket: } R|\beta\rangle \iff \text{bra: } \langle \beta | R^\dagger.
\]

In terms of its matrix representation, we have

\[
\langle j | R^\dagger | i \rangle = \langle i | R | j \rangle^* = R^*_ij.
\]

**The matrix for \( R^\dagger \) is the conjugate transpose of the matrix for \( R \).**
Adjoint Operators

Example

Let $R = |0\rangle\langle\tilde{+}| + |\tilde{-}\rangle\langle1|$, where $|\tilde{\pm}\rangle = \sqrt{1/2}(|0\rangle \pm i|1\rangle)$. What is $R^\dagger$?
Eigenvalues and Eigenvectors

The eigenvalue of a linear operator \( R \) is a complex number \( \lambda \) such that \( R|\psi\rangle = \lambda|\psi\rangle \) for some vector \( |\psi\rangle \).

For a fixed eigenvalue \( \lambda \), the set \( V_\lambda = \{|\varphi\rangle : R|\varphi\rangle = \lambda|\varphi\rangle \} \) forms a vector space. This vector space \( V_\lambda \) is called the eigenspace associated with \( \lambda \) (or the \( \lambda \)-eigenspace), and any vector from this space is an eigenvector of \( R \) with eigenvalue \( \lambda \).

The kernel (or null space) of an operator \( R \) is the eigenspace associated with the eigenvalue 0, and it is denoted by \( ker(R) \).

The rank of an operator \( R : \mathcal{H}_1 \to \mathcal{H}_2 \), denoted \( rk(R) \) is the number given

\[
rk(R) = \dim(\mathcal{H}_1) - \dim(ker(R)).
\]
Important Classes of Operators

The Picture:
Projections

A **projection** on a $d$-dimensional space $\mathcal{H}$ is any operator of the form

$$P_V = \sum_{i=1}^{s} |\epsilon_i\rangle\langle\epsilon_i|,$$

where the $\{|\epsilon_i\rangle\}_{i=1}^{s}$ is an orthonormal set of vectors for some integer $1 \leq s \leq d$.

The $\{|\epsilon_i\rangle\}_{i=1}^{s}$ form a basis for some $s$-dimensional vector subspace $V \subset \mathcal{H}$, and $P_V$ is called the **subspace projector** onto $V$:

$$P_V|\psi\rangle \in V \quad \text{for any } |\psi\rangle \in \mathcal{H}.$$
Projections

**Example:** For any normalized vector $|\psi\rangle$, the operator $|\psi\rangle\langle\psi|$ is a rank-one projector.

**Example:** Prove that every projector satisfies $P^2 = P$.

**Example:** Prove that 0 and 1 are the only possible eigenvalues for a projector.
Normal Operators

An operator $N \in L(\mathcal{H})$ is called **normal** if $N^\dagger N = NN^\dagger$.

**Lemma:** If $N \in L(\mathcal{H})$ is normal with distinct eigenvalues $\lambda_1$ and $\lambda_2$, then the associated eigenspaces $V_{\lambda_1}$ and $V_{\lambda_2}$ are orthogonal. Furthermore, $\mathcal{H} = \bigcup_{\lambda_i} V_{\lambda_i}$, where the union is taken over all distinct eigenvalues of $N$.

The eigenspace decomposition picture:
Normal Operators: The Spectral Decomposition

**Spectral Decomposition**

**Theorem:** Every normal operator \( N \in L(\mathcal{H}) \) can be uniquely written as

\[
N = \sum_{i=1}^{n} \lambda_i P_{\lambda_i},
\]

where the \( \{\lambda_i\}_{i=1}^{n} \) are the distinct eigenvalues of \( N \) and the \( \{P_{\lambda_i}\}_{i=1}^{n} \) are the corresponding eigenspace projectors.

We can further decompose the eigenspace projectors and write:

\[
N = \sum_{k=1}^{\dim(\mathcal{H})} \lambda_k |\lambda_k\rangle\langle\lambda_k|.
\]

Here the \( \lambda_k \) are not necessarily distinct.
Unitary Operators

An operator \( U \in \mathcal{L}(\mathcal{H}) \) is called **unitary** if \( U^\dagger U = UU^\dagger = \mathbb{I} \).

Example

For arbitrary \( \alpha, \beta, \gamma \in \mathbb{R} \) the operator \( U(\alpha, \beta, \gamma) \) on \( \mathbb{C}^2 \) given by

\[
U(\alpha, \beta, \gamma) = \begin{pmatrix}
  e^{-i(\alpha+\gamma)/2} \cos \beta/2 & -e^{-i(\alpha-\gamma)/2} \sin \beta/2 \\
  e^{-i(-\alpha+\gamma)/2} \sin \beta/2 & e^{i(\alpha+\gamma)/2} \cos \beta/2
\end{pmatrix}
\]

is unitary.
Unitary Operators

Unitary operators preserve inner products: For vectors $U|\psi\rangle$ and $U|\phi\rangle$, their inner product is

$$\langle \psi | U^\dagger U | \phi \rangle = \langle \psi | I | \phi \rangle = \langle \psi | \phi \rangle.$$

If $\{|\epsilon_i\rangle\}_{i=1}^d$ and $\{|\delta_i\rangle\}_{i=1}^d$ are orthonormal bases for $\mathcal{H}$, then

$$U = \sum_{i=1}^d |\epsilon_i\rangle\langle \delta_i|$$

is a unitary operator transforming one basis to another.

$$U^\dagger U = \sum_{i=1}^d |\delta_i\rangle\langle \epsilon_i| \sum_{j=1}^d |\epsilon_j\rangle\langle \delta_j| = \sum_{i=1}^d |\delta_i\rangle\langle \delta_i| = I.$$

The spectral decomposition can be written as $N = U\Lambda U^\dagger$, where $\Lambda$ is a diagonal matrix in the computational basis with the eigenvalues of $N$ along the diagonal.
Hermitian and Positive Operators

An operator $A$ is called **hermitian** (or self-adjoint) if $A^\dagger = A$.

**Fact:** Every hermitian operator has real eigenvalues.

**Proof:**

A **positive operator** is any normal operator $A$ with non-negative eigenvalues.

Equivalently, $A$ is positive iff $\langle \psi | A | \psi \rangle \geq 0$ for all $|\psi\rangle \in \mathcal{H}$.

For any operator $R \in \mathbb{L}(\mathcal{H}_1, \mathcal{H}_2)$, the operators $R^\dagger R$ and $RR^\dagger$ are positive.
Functions of Normal Operators

The spectral decomposition allows us to define functions of operators. Notice that

\[ N = \sum_{k=1}^{n} \lambda_k P_{\lambda_k} \quad \Rightarrow \quad N^m = \sum_{k=1}^{n} \lambda_k^m P_{\lambda_k} \quad \text{for all } m = 1, \cdots. \]

Then for any complex function \( f(z) \) with Taylor expansion

\[ f(z) = \sum_{k=0}^{\infty} \alpha_k z^k, \]

we define the operator function \( \hat{f}(N) := \sum_{k=0}^{\infty} \alpha_k N^k. \)

Consequently, for a function \( f : \mathcal{X} \rightarrow \mathbb{C} \), if \( N \) has eigenvalues lying in \( \mathcal{X} \), then we can define a new operator

\[ \hat{f}(N) = \sum_{i=1}^{n} f(\lambda_i) P_{\lambda_i}. \]
Functions of Normal Operators

**Example:** For the function \( f(z) = e^z \), we have

\[
\hat{f}(N) = e^N = \sum_{i=1}^{n} e^{\lambda_i} P_{\lambda_i}.
\]

For functions \( f(z) \) not defined at \( z = 0 \), define \( \hat{f}(N) = \sum_{\lambda_i \neq 0} f(\lambda_i) P_{\lambda_i} \).

**Example:** For the multiplicative inverse \( f(z) = z^{-1} = \frac{1}{z} \), define \( N^{-1} := \sum_{\lambda_i \neq 0} \lambda_i^{-1} P_{\lambda_i} \). Note that

\[
N^{-1} N = NN^{-1} = \sum_{\lambda_i \neq 0} \lambda_i P_{\lambda_i} \sum_{\lambda_j \neq 0} \lambda_j^{-1} P_{\lambda_j} = \sum_{\lambda_i \neq 0} P_{\lambda_i} = P_{\text{supp}(N)}.
\]

where \( \text{supp}(N) := \bigcup_{\lambda_i \neq 0} V_{\lambda_i} \) is the **support** of \( N \).
Singular Value Decomposition

**Theorem:** Every operator $R \in L(\mathcal{H})$ can be written as

$$R = \sum_{k=1}^{n} \sigma_k U P_{\sigma_k},$$

where $\{\sigma_k\}_{k=1}^{n}$ are the distinct singular values of $R$, $U$ is a unitary operator, and the $\{P_{\sigma_k}\}_{k=1}^{n}$ are projections on the eigenspaces of $R^\dagger R$.

Equivalently, $R$ can be written as

$$R = V \Lambda_\sigma W^\dagger,$$

where $V$ and $W$ are unitaries, and $\Lambda_\sigma$ is a diagonal matrix in the computational basis with diagonal elements being the singular values of $R$. 