Linear Algebra using Dirac Notation: Pt. 3

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Functions of Normal Operators

**Example:** For the function \( f(z) = e^z \), we have

\[
\hat{f}(N) = e^N = \sum_{i=1}^{n} e^{\lambda_i} P_{\lambda_i}.
\]

For functions \( f(z) \) not defined at \( z = 0 \), define \( \hat{f}(N) = \sum_{\lambda_i \neq 0} f(\lambda_i)P_{\lambda_i} \).

**Example:** For the multiplicative inverse \( f(z) = z^{-1} = \frac{1}{z} \), define \( N^{-1} := \sum_{\lambda_i \neq 0} \lambda_i^{-1} P_{\lambda_i} \). Note that

\[
N^{-1}N = NN^{-1} = \sum_{\lambda_i \neq 0} \lambda_i P_{\lambda_i} \sum_{\lambda_j \neq 0} \lambda_j^{-1} P_{\lambda_j} = \sum_{\lambda_i \neq 0} P_{\lambda_i} = P_{\text{supp}(N)}.
\]

where \( \text{supp}(N) := \bigcup_{\lambda_i \neq 0} V_{\lambda_i} \) is the **support** of \( N \).
Theorem: Every operator $R \in L(\mathcal{H})$ can be written as

$$R = \sum_{k=1}^{n} \sigma_k U P_{\sigma_k},$$

where $\{\sigma_k\}_{k=1}^{n}$ are the distinct singular values of $R$, $U$ is a unitary operator, and the $\{P_{\sigma_k}\}_{k=1}^{n}$ are projections on the eigenspaces of $R^\dagger R$.

Equivalently, $R$ can be written as

$$R = V \Lambda_\sigma W^\dagger,$$

where $V$ and $W$ are unitaries, and $\Lambda_\sigma$ is a diagonal matrix in the computational basis with diagonal elements being the singular values of $R$. 
If $\mathcal{H}^A$ is a Hilbert space with computational basis $\{|i\rangle^A\}_{i=1}^{d_A}$ and $\mathcal{H}^B$ is a Hilbert space with computational basis $\{|i\rangle^B\}_{i=1}^{d_B}$, then their tensor product space is a Hilbert space $\mathcal{H}^A \otimes \mathcal{H}^B$ with orthonormal basis given by $\{|i\rangle^A \otimes |j\rangle^B\}_{i=1,j=1}^{d_A,d_B}$, called a tensor product basis.

Every vector $|\psi\rangle^{AB} \in \mathcal{H}^A \otimes \mathcal{H}^B$ can be written as a linear combination of the basis vectors:

$$|\psi\rangle^{AB} = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} c_{ij} |i\rangle^A \otimes |j\rangle^B.$$
Tensor Products

The tensor product is a map \( \otimes : \mathcal{H}^A \times \mathcal{H}^B \rightarrow \mathcal{H}^A \otimes \mathcal{H}^B \) that generates the tensor product vector \( |\psi\rangle^A \otimes |\phi\rangle^B \in \mathcal{H}^A \otimes \mathcal{H}^B \) for \( |\psi\rangle^A \in \mathcal{H}^A \) and \( |\phi\rangle^B \in \mathcal{H}^B \).

The tensor product is a bilinear operation meaning that it satisfies the three properties:

1. \((c|\psi\rangle^A) \otimes |\phi\rangle^B = |\psi\rangle^A \otimes (c|\phi\rangle^B) = c(|\psi\rangle^A \otimes |\phi\rangle^B)\)
   \(\forall |\psi\rangle^A \in \mathcal{H}^A, |\phi\rangle^B \in \mathcal{H}^B, c \in \mathbb{C},\)

2. \((|\psi\rangle^A + |\omega\rangle^A) \otimes |\phi\rangle^B = |\psi\rangle^A \otimes |\phi\rangle^B + |\omega\rangle^A \otimes |\phi\rangle^B\)
   \(\forall |\psi\rangle^A, |\omega\rangle^A \in \mathcal{H}^A, |\phi\rangle^B \in \mathcal{H}^B,\)

3. \(|\psi\rangle^A \otimes (|\phi\rangle^B + |\omega\rangle^B) = |\psi\rangle^A \otimes |\phi\rangle^B + |\psi\rangle^A \otimes |\omega\rangle^B\)
   \(\forall |\psi\rangle^A \in \mathcal{H}^A, |\phi\rangle^B, |\omega\rangle^B \in \mathcal{H}^B.\)
Tensor Products

Any basis \( \{|i'\rangle^A\}_{i=1}^{d_A} \) for \( \mathcal{H}^A \) and basis \( \{|i'\rangle^B\}_{i=1}^{d_B} \) for \( \mathcal{H}^B \) can be “tensored” together to form a basis \( \{|i'\rangle^A \otimes |j'\rangle^B\}_{i,j=1}^{d_A,d_B} \) for \( \mathcal{H}^A \otimes \mathcal{H}^B \).

Thus, we can write any \( |\psi\rangle^{AB} \in \mathcal{H}^A \otimes \mathcal{H}^B \) as

\[
|\psi\rangle^{AB} = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} c_{ij} |i\rangle^A \otimes |j\rangle^B = \sum_{i=1}^{d_A} \sum_{j=1}^{d_B} c'_{ij} |i'\rangle^A \otimes |j'\rangle^B.
\]

The \( c_{ij} \) and \( c'_{ij} \) are different coefficients for \( |\psi\rangle^{AB} \) in the different bases.

We drop the superscript labels \( A \) and \( B \) on the kets when the Hilbert spaces are clear.

Also, \( |\psi\rangle \otimes |\phi\rangle \) is often written as \( |\psi\rangle |\phi\rangle \), or just \( |\psi\phi\rangle \).
Example

Suppose $|\psi\rangle = a |0\rangle + b |1\rangle$ and $|\phi\rangle = c |0\rangle + d |1\rangle$.

(a) Express $|\psi\rangle \otimes |\phi\rangle$ in the tensor product basis \{ $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ \}.

(b) Express $|\phi\rangle \otimes |\psi\rangle$ in the tensor product basis \{ $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ \}.

(c) Express $|\psi\rangle \otimes |\phi\rangle$ in the tensor product basis \{ $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ \}, where $|\pm\rangle = \sqrt{1/2}(|0\rangle \pm |1\rangle)$. 
Remark

The tensor product space $\mathcal{H}^A \otimes \mathcal{H}^B$ is much larger than the set of tensor product vectors!

\[ S = \{ |\alpha\rangle \otimes |\beta\rangle : |\alpha\rangle \in \mathcal{H}^A, \ |\beta\rangle \in \mathcal{H}^B \} \]

\[ \mathcal{H}^A \otimes \mathcal{H}^B = \text{span}(S) \]

Not every $|\psi\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B$ is the tensor product of two vectors:

$|\psi\rangle \neq |\alpha\rangle \otimes |\beta\rangle$.

As we will see, such vectors $|\psi\rangle$ correspond to entangled states in quantum mechanics.
Tensor Products

Definition (Tensor Product of Operators)

If $A$ is an operator on $\mathcal{H}^A$ and $B$ is an operator on $\mathcal{H}^B$, then their tensor product $A \otimes B$ is a linear operator on $\mathcal{H}^A \otimes \mathcal{H}^B$ with action defined by

$$A \otimes B \left( \sum_{i,j} c_{ij} |i\rangle \otimes |j\rangle \right) = \sum_{i,j} c_{ij} A|i\rangle \otimes B|j\rangle.$$ 

If $A$ and $C$ are operators on $\mathcal{H}^A$, and $B$ and $D$ are operators on $\mathcal{H}^B$, then

1. $A \otimes B + C \otimes D$ is an operator on $\mathcal{H}^A \otimes \mathcal{H}^B$ with action

   $$(A \otimes B + C \otimes D)|\psi\rangle = A \otimes B|\psi\rangle + C \otimes D|\psi\rangle \quad \forall |\psi\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B,$$

2. $(A \otimes B)(C \otimes D) = AC \otimes BD$. 
Tensor Products

Example: The partial contraction of $|\phi\rangle$ on system $A$ is the operator $\langle \phi |^A \otimes I^B \in \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B, \mathcal{H}^B)$ such that for any $|\psi\rangle^{AB} = \sum_{i,j} c_{ij} |i\rangle^A |j\rangle^B$,

$$\langle \phi |^A \otimes I^B (|\psi\rangle^{AB}) = \langle \phi |^A \otimes I^B \left( \sum_{i,j} c_{ij} |i\rangle^A |j\rangle^B \right) = \sum_{ij} c_{ij} \langle \phi | i \rangle |j\rangle^B.$$ 

Example: If $|\gamma\rangle^{AB} = \sum_{i,j} a_{i,j} |i\rangle \otimes |j\rangle$ and $|\omega\rangle^{AB} = \sum_{i,j} b_{i,j} |i\rangle \otimes |j\rangle$ are two vectors in $\mathcal{H}^A \otimes \mathcal{H}^B$, their inner product is a full contraction:

$$\langle \gamma | \omega \rangle = \left( \sum_{i,j} a_{i,j}^* \langle i |^A \otimes \langle j |^B \right) \left( \sum_{k,l} b_{k,l} |k\rangle^A \otimes |l\rangle^B \right)$$

$$= \sum_{i,j,k,l} a_{i,j}^* b_{k,l} \langle i | k \rangle^A \langle j | l \rangle^B = \sum_{i,j} a_{i,j}^* b_{i.j}.$$
Tensor Products

Recall any $A \in \mathcal{L}(\mathcal{H}^A, \mathcal{H}^{A'})$ and $B \in \mathcal{L}(\mathcal{H}^B, \mathcal{H}^{B'})$ can be expressed as

$$A = \sum_{i=1}^{d_A} \sum_{k=1}^{d_{A'}} a_{ik} |i\rangle \langle k|,$$

$$B = \sum_{j=1}^{d_B} \sum_{l=1}^{d_{B'}} b_{jl} |j\rangle \langle l|.$$

Likewise, any operator $K \in \mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B, \mathcal{H}^{A'} \otimes \mathcal{H}^{B'})$ can be expressed as

$$K = \sum_{i=1}^{d_A} \sum_{k=1}^{d_{A'}} \sum_{j=1}^{d_B} \sum_{l=1}^{d_{B'}} c_{ij,kl} |i\rangle \langle k| \otimes |j\rangle \langle l|.$$

Remark

$\mathcal{L}(\mathcal{H}^A \otimes \mathcal{H}^B, \mathcal{H}^{A'} \otimes \mathcal{H}^{B'})$ is much larger than the set of tensor product operators!
Tensor Products

Example

Consider space $\mathcal{H}^A \otimes \mathcal{H}^B \equiv \mathbb{C}^2 \otimes \mathbb{C}^2$, and let $\sigma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. What is the action of $\sigma_x \otimes \sigma_x$ on an arbitrary $|\psi\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B$?

Example

Consider the operator $F = |00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|$. What is its action on an arbitrary $|\psi\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B$? The operator $F$ is called the SWAP operator.
Tensor Products

The matrix representation of tensor products is given in terms of the Kronecker matrix product.

\[
A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1d_1} \\
  a_{21} & a_{22} & \cdots & a_{2d_1} \\
  \vdots  \\
  a_{d_11} & \cdots & \cdots & a_{d_1d_1}
\end{pmatrix}, \quad B = \begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1d_2} \\
  b_{21} & b_{22} & \cdots & b_{2d_2} \\
  \vdots  \\
  b_{d_21} & \cdots & \cdots & b_{d_2d_2}
\end{pmatrix},
\]

\[
A \otimes B = \begin{pmatrix}
  a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{12}b_{11} & a_{12}b_{12} & \cdots & a_{1d_1}b_{1d_2} \\
  a_{11}b_{21} & a_{11}b_{22} & \cdots & a_{12}b_{11} & a_{12}b_{12} & \cdots & a_{1d_1}b_{2d_2} \\
  \vdots  \\
  a_{21}b_{11} & a_{21}b_{12} & \cdots & a_{22}b_{11} & a_{22}b_{12} & \cdots & a_{2d_1}b_{1d_2} \\
  \vdots  \\
  a_{d_11}b_{d_21} & a_{d_11}b_{d_22} & \cdots & a_{d_12}b_{d_21} & a_{d_12}b_{d_22} & \cdots & a_{d_1d_1}b_{d_2d_2}
\end{pmatrix}
\]
Tensor Products

Example

For $|\psi\rangle = a|0\rangle + b|1\rangle$ and $|\phi\rangle = c|0\rangle + d|1\rangle$, what is the matrix representation of $|\psi\rangle \otimes |\phi\rangle$ in the computational basis? What is the matrix representation of $\langle\psi| \otimes \langle\phi|$?

Example

What is the matrix representation of $\sigma_x \otimes \sigma_x$?

Example

What is the matrix representation of $F$?