Midterm Review Problems

As usual, we let

\[ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(1) Give the matrix representation of the following operator

\[ |0\rangle \langle 0| \otimes (\sigma_x + \sigma_z) + (I + \sigma_y) \otimes |1\rangle \langle 1|. \]

(2) The ensemble \( \mathcal{E} = \{ (|0\rangle, \frac{3}{4}), (|1\rangle, \frac{1}{4}) \} \) has ensemble average

\[ \rho = \frac{3}{4} |0\rangle \langle 0| + \frac{1}{4} |1\rangle \langle 1|. \]

(i) Give a purification of \( \rho \).

(ii) Give another ensemble \( \mathcal{E}' \) that has the same ensemble average \( \rho \).

(iii) Suppose that a measurement is performed on a system in the mixed state \( \rho \) described by the observable

\[ O = r_x \sigma_x + r_y \sigma_y + r_z \sigma_z, \]  

where \( r_x, r_y, r_z \) are real numbers. Compute the expectation value of this measurement as a function of the \( r_x, r_y, r_z \).

(iv) Let \( |\Psi\rangle^{AB} \) be a purification of \( \rho \). What is the expectation value when observable \( O \otimes I \) is measured on systems \( A \) and \( B \) in state \( |\Psi\rangle^{AB} \).

(3) Suppose that Alice and Bob share the entangled state \( |\psi\rangle = \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle + |22\rangle) \). Consider a quantum observable \( O = \lambda_0 |\phi\rangle \langle \phi| + \lambda_1 (I_3 - |\phi\rangle \langle \phi|) \) where \( |\phi\rangle = \frac{1}{\sqrt{3}} (|0\rangle + e^{2\pi/3}|1\rangle + e^{4\pi/3}|2\rangle) \) and each of the \( \lambda_i \) are distinct. Alice and Bob each measure their subsystem with a measurement described by this observable.

What are the possible post-measurement states, and with what probability are they each obtained?