Some examples of quantum systems:

1. Dual-Rail photonic systems;
2. Photon polarization;

Our goal is to mathematically describe these systems using the language of linear algebra.

The framework for this description is built on four axioms/postulates:

1. The State Space Postulate;
2. The Evolution Postulate;
3. The Composition of Systems Postulate;
The State Space Postulate

Every quantum system is represented by a Hilbert space $\mathcal{H}$. Each physical state of the system is represented by a normalized vector in $\mathcal{H}$, and every normalized vector in $\mathcal{H}$ corresponds to a physical state of the system.

Experimental observation determines what type of Hilbert space is associated with a given physical system.

For example, electrons are experimentally observed to possess an intrinsic degree of freedom known as “spin.” This is a two-level degree of freedom. Therefore, the spin of an electron is described by a two-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^2$. It is a qubit system.

An electron spin state is described by the vector $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$ and $\{|0\rangle, |1\rangle\}$ is some chosen basis for the system.
The State Space Postulate

Global Phase

Two normalized vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ in $\mathcal{H}$ differ by a global phase if

$$|\psi_2\rangle = e^{i\varphi} |\psi_1\rangle \quad \varphi \in (0, 2\pi).$$

They represent the same physical state.

Relative Phase

Two normalized vectors $|\psi_1\rangle$ and $|\psi_2\rangle$ in $\mathcal{H}$ differ by a relative phases if

$$|\psi_1\rangle = \sum_{i=0}^{d-1} c_i |i\rangle, \quad |\psi_2\rangle = \sum_{i=0}^{d-1} c_i e^{i\varphi_i} |i\rangle \quad \varphi_i \neq 0 \text{ for some } i.$$

They represent different physical states.
The Evolution Postulate

The time-evolution of a closed quantum system is described by a unitary operator. That is, it is possible for a closed system to evolve from state $|\psi_0\rangle$ at time $t_0$ to state $|\psi_1\rangle$ at time $t_1$ if and only if there exists a unitary operator $U$ such that

$$|\psi_1\rangle = U|\psi_0\rangle.$$ 

The actual evolution of a system is determined by its Hamiltonian $H$ and Schrödinger’s Equation: $i\hbar \frac{d}{dt} |\psi\rangle = H|\psi\rangle$.

For time independent $H$, the solution is $|\psi_1\rangle = e^{-iH(t_1-t_0)/\hbar} |\psi_0\rangle = U|\psi_0\rangle$.

The evolution is unitary:

$$UU^\dagger = e^{-iH(t_1-t_0)/\hbar} e^{iH^\dagger(t_1-t_0)/\hbar} = \mathbb{I}$$

since $H^\dagger = H$. 

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Qubits

Every qubit state has the form $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$.

Without loss of generality we can assume that $\alpha$ is real. Since if $\alpha = re^{i\kappa}$ with $r \geq 0$ and $\kappa \in [0, 2\pi)$, we can multiply $|\psi\rangle$ by a global phase $e^{-i\kappa}$ and consider the equivalent state vector $e^{-i\kappa}|\psi\rangle = r|0\rangle + \beta e^{-i\kappa}|1\rangle$.

Therefore, a general qubit state can be written as

$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle \quad \theta \in [0, \pi); \phi \in [0, 2\pi).$$

Why the $\theta/2$ instead of just $\theta$?
So we can easily establish a one-to-one correspondence between qubit states and points on the three-dimensional unit sphere, called the **Bloch sphere**.
Qubits

The Bloch Sphere

The state \( |\hat{r}\rangle := |\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle \) is represented by the three-dimensional unit vector \( \hat{r} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \), called the **Bloch vector** of \( |\psi\rangle \). Diagram:
Qubits

The **Pauli operators** \( \{ \sigma_x, \sigma_y, \sigma_z \} \) are three operators defined in terms of their action on the computational basis. Their matrix representations in the computational basis are

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

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<tr>
<th>Operator</th>
<th>Eigenvalues</th>
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Qubits

The Pauli $\sigma_x$ and $\sigma_z$ are transformed from one to another using the **Hadamard operator** $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$:

\[
H\sigma_x H = \sigma_z, \quad H\sigma_z H = \sigma_x.
\]

Note that $H\sigma_y H = -\sigma_y$.

**Example**

The state $|\hat{r}\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle$ has Bloch vector lying in the $x - z$ plane of the Bloch sphere, the so-called “real plane.” How does it transform under $H$?
Qubits

A general rotation on the Bloch sphere is described by a $3 \times 3$ real orthogonal matrix having determinant $+1$. The group of all such rotations is called the **Special Orthogonal Group**, denoted by $SO(3)$.

Every operator in $SO(3)$ is characterized by an axis of rotation $\hat{n}$ and a rotation angle $\theta$. The operator $O_{\hat{n}}(\theta)$ rotates any unit vector $\hat{r}$ about axis $\hat{n}$ by angle $\theta$ to generate the new vector $\hat{r}' = O_{\hat{n}}(\theta)\hat{r}$.

For any $O_{\hat{n}}(\theta) \in SO(3)$ that rotates

$$O_{\hat{n}}(\theta) : \hat{r} \rightarrow \hat{r}' = O_{\hat{n}}(\theta)\hat{r},$$

there is a corresponding $2 \times 2$ unitary matrix $R_{\hat{n}}(\theta)$ that transforms state vectors

$$R_{\hat{n}}(\theta) : |\hat{r}\rangle \rightarrow |\hat{r}'\rangle = |O_{\hat{n}}(\theta)\hat{r}\rangle.$$
Qubits

A general rotation of state vectors can be constructed using three basic rotations, each of these being generated by one of the Pauli operators:

\[
R_x(\theta) := e^{-i\theta\sigma_x/2} = \cos(\theta/2)\mathbb{I} - i\sin(\theta/2)\sigma_x = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}
\]

\[
R_y(\theta) := e^{-i\theta\sigma_y/2} = \cos(\theta/2)\mathbb{I} - i\sin(\theta/2)\sigma_y = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}
\]

\[
R_z(\theta) := e^{-i\theta\sigma_z/2} = \cos(\theta/2)\mathbb{I} - i\sin(\theta/2)\sigma_z = \begin{pmatrix} \cos(\theta/2) & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}
\]

For a general unit vector \( \hat{n} = (n_x, n_y, n_z) \), let \( \vec{\sigma} := \sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z} \), and

\[
R_{\hat{n}}(\theta) := e^{-i\theta\hat{n}\cdot\vec{\sigma}/2} = \cos(\theta/2)\mathbb{I} - i\sin(\theta/2) (\hat{n} \cdot \vec{\sigma})
\]
Qubits

Notice that $R_{\hat{n}}(\theta)$ is a $2 \times 2$ unitary matrix with determinant $+1$.

Interpretation

When $O_{\hat{n}}(\theta) : \hat{r} \rightarrow \hat{r}'$ rotates Bloch vectors, the operator $R_{\hat{n}}(\theta)$ rotates the corresponding state vectors $|\hat{r}\rangle \rightarrow |\hat{r}'\rangle$.

To see this, consider an arbitrary $|\hat{r}\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle$. Note:

$$
|\hat{r}\rangle\langle\hat{r}| := \cos^2(\theta/2)|0\rangle\langle0| + \sin^2(\theta/2)|1\rangle\langle1|
+ \sin(\theta/2)\cos(\theta/2)(e^{-i\phi}|0\rangle\langle1| + e^{i\phi}|1\rangle\langle0|)
= \frac{1}{2} (I + \cos \theta \sigma_z + \sin \theta \sin \phi \sigma_y + \sin \theta \cos \phi \sigma_x)
= \frac{1}{2} (I + \hat{r} \cdot \vec{\sigma}).
$$
Qubits

\[ R_{\hat{n}}(\theta) (|\hat{r}\rangle \langle \hat{r}|) R_{\hat{n}}(\theta)^\dagger \]
\[
= \frac{1}{2} \left[ \cos(\theta/2) \mathbb{I} - i \sin(\theta/2) (\hat{n} \cdot \vec{\sigma}) \right] \left[ \mathbb{I} + \hat{r} \cdot \vec{\sigma} \right] \left[ \cos(\theta/2) \mathbb{I} + i \sin(\theta/2) (\hat{n} \cdot \vec{\sigma}) \right] \\
= \frac{1}{2} \left[ \mathbb{I} + \cos^2(\theta/2) \hat{r} \cdot \vec{\sigma} + \sin^2(\theta/2) (\hat{n} \cdot \vec{\sigma})(\hat{r} \cdot \vec{\sigma})(\hat{n} \cdot \vec{\sigma}) \right. \\
\left. + i \cos(\theta/2) \sin(\theta/2) [\hat{r} \cdot \vec{\sigma}, \hat{n} \cdot \vec{\sigma}] \right] \\
= \frac{1}{2} \left[ \mathbb{I} + (\cos^2(\theta/2) - \sin^2(\theta/2)) \hat{r} \cdot \vec{\sigma} + 2 \sin^2(\theta/2) (\hat{n} \cdot \hat{r}) \hat{n} \cdot \vec{\sigma} \right. \\
\left. + 2 \cos(\theta/2) \sin(\theta/2) (\hat{n} \times \hat{r}) \cdot \vec{\sigma} \right] \\
= \frac{1}{2} \left[ \mathbb{I} + \cos \theta \hat{r} \cdot \vec{\sigma} + (1 - \cos \theta)(\hat{n} \cdot \hat{r}) \hat{n} \cdot \vec{\sigma} + \sin \theta (\hat{n} \times \hat{r}) \cdot \vec{\sigma} \right],
\]

where we have used the relationship: \((\hat{r} \cdot \vec{\sigma})(\hat{n} \cdot \vec{\sigma}) = (\hat{n} \cdot \hat{r}) \mathbb{I} + i(\hat{r} \times \hat{n}) \cdot \vec{\sigma}\).
Qubits

Rodrigues’ Rotation Formula

When vector $\hat{r}$ is rotated about axis $\hat{n}$ by angle $\theta$, the resultant vector is

$$\hat{r}' = \hat{r} \cos \theta + (\hat{n} \times \hat{r}) \sin \theta + \hat{n}(\hat{n} \cdot \hat{r})(1 - \cos \theta).$$

Compare:

$$R_{\hat{n}}(\theta) (|\hat{r}\rangle \langle \hat{r}|) R_{\hat{n}}(\theta)\dagger = \frac{1}{2}(\mathbb{I} + [\cos \theta \hat{r} + (1 - \cos \theta)(\hat{n} \cdot \hat{r})\hat{n} + \sin \theta(\hat{n} \times \hat{r})] \cdot \vec{\sigma})$$

$$= \frac{1}{2}(\mathbb{I} + \hat{r}' \cdot \vec{\sigma}).$$

Therefore, $R_{\hat{n}}(\theta)$ rotates the state $|\hat{r}\rangle$ to $|\hat{r}'\rangle$. 
Qubits

Recall that $R_{\hat{n}}(\theta)$ is a unitary matrix with determinant one. The group of all such unitaries is called the **special unitary group**, denoted by $SU(2)$.

In fact, every $U \in SU(2)$ has the form $\pm R_{\hat{n}}(\theta)$ for some unit vector $\hat{n}$ and angle $\theta$.

**Summary**

There is a one-to-two correspondence $O_{\hat{n}}(\theta) \leftrightarrow \pm R_{\hat{n}}(\theta)$ between $SO(3)$ and $SU(2)$ such that

$$[O_{\hat{n}}(\theta)\hat{\hat{a}}] \cdot \sigma = (\pm R_{\hat{n}}(\theta))(\hat{\hat{a}} \cdot \vec{\sigma})(\pm R_{\hat{n}}(\theta))^\dagger \quad \forall \hat{\hat{a}}.$$  

**Second Isomorphism Theorem**

$$SO(3) \cong SU(2) \setminus \mathbb{Z}_2.$$