Principles of Quantum Mechanics Pt. 2

PHYS 500 - Southern Illinois University

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The Composition of Systems Postulate

If $A$ and $B$ are two quantum systems with state spaces $\mathcal{H}^A$ and $\mathcal{H}^B$ respectively, the state space of the their combined physical system is the tensor product space $\mathcal{H}^A \otimes \mathcal{H}^B$. This is often called a **bipartite** system.

The bipartite product state $|\psi_1^A\rangle|\psi_2^B\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B$ describes system $A$ in state $|\psi_1^A\rangle$ and system $B$ in state $|\psi_2^B\rangle$.

But $\mathcal{H}^A \otimes \mathcal{H}^B$ is much larger than the set of tensor product states.

A bipartite state $|\psi^{AB}\rangle$ is **entangled** if it is not a tensor product state; i.e. $|\psi^{AB}\rangle \neq |\psi_1^A\rangle|\psi_2^B\rangle$. A non-entangled state is also called a **separable** state. In an entangled state, no state vector can be assigned to the individual subsystems.
Entanglement

How to determine if a state $|\Psi\rangle^{AB}$ is entangled?

For two qubits, expand $|\Psi\rangle^{AB}$ in the computational basis

$$|\Psi\rangle^{AB} = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle,$$

and suppose that

$$|\Psi\rangle^{AB} = |\psi_1\rangle^A \otimes |\psi_2\rangle^B = (x_0|0\rangle + x_1|1\rangle) \otimes (y_0|0\rangle + y_1|1\rangle)$$

$$= x_0y_0|00\rangle + x_0y_1|01\rangle + x_1y_0|10\rangle + x_1y_1|11\rangle.$$

Comparing coefficients, $|\Psi\rangle^{AB}$ is a product state if and only if $ad = bc$.

Proof:
Entanglement

A more elegant method for deciding entanglement

Recall the isomorphism $\mathcal{H}^A \otimes \mathcal{H}^B \cong \mathcal{L}(\mathcal{H}^A, \mathcal{H}^B)$ given by

$$
\sum_{i,j=1}^{d_A,d_B} c_{ij} |i\rangle^A |j\rangle^B \iff \sum_{i,j=1}^{d_A,d_B} c_{ij} |i\rangle^A \langle j|^B.
$$

For product states:

$$
|\psi_1\rangle|\psi_2\rangle = \sum_{i,j=1}^{d_A,d_B} (x_i |i\rangle) (y_j |j\rangle) \iff \sum_{i,j=1}^{d_A,d_B} (x_i |i\rangle) (y_j \langle j|) = |\psi_1\rangle\langle \psi_2^*|
$$

where $|\psi_2^*\rangle := \sum_{j=1}^{d_B} y_j^* |j\rangle$. The operator $|\psi_1\rangle\langle \psi_2^*|$ is rank one!
Entanglement

To decide entanglement:

1. For a bipartite state $|\Psi\rangle^{AB}$, expand in the computational product basis and obtain its corresponding operator $T_{\Psi}$ via the mapping

$$|\Psi\rangle^{AB} = \sum_{i,j=1}^{d_A,d_B} c_{ij} |i\rangle^A |j\rangle^B \quad \Leftrightarrow \quad T_{\Psi} = \sum_{i,j=1}^{d_A,d_B} c_{ij} |i\rangle^A \langle j|^B.$$ 

2. Write out $T_{\Psi}$ in matrix form and compute its rank.

3. Recall that a matrix has rank $r$ if and only if all of its $(r+1)$-minors vanish, and there exists at least one nonvanishing $r$-minor

4. $|\Psi\rangle^{AB}$ is entangled if and only if $T_{\Psi}$ is rank one.
Entanglement

Example

Is the state

$$|\psi\rangle^{AB} = a|00\rangle + \sqrt{ab}|01\rangle + c|02\rangle + \sqrt{ab}|10\rangle + b|11\rangle + c\sqrt{b/a}|12\rangle - c|22\rangle$$

entangled?
Unitary Evolution on Composite Systems

By the second postulate, a closed bipartite system will evolve unitarily. That is, $|\psi_0\rangle^{AB} \rightarrow |\psi_1\rangle^{AB} = U|\psi_0\rangle^{AB}$. The operator $U$ is sometimes called a **gate**.

Any tensor product unitary $U = U^A \otimes U^B$ is called a **local unitary** (LU).

Two states $|\Psi\rangle$ and $|\Psi'\rangle$ are called **LU equivalent**, denoted by $|\Psi\rangle \overset{LU}{\equiv} |\Psi'\rangle$, if there exists a local unitaries $U \otimes V$ such that $|\Psi\rangle = U \otimes V|\Psi'\rangle$.

**Example**

Let $|\Psi\rangle = \sqrt{\lambda_1}|a_1\rangle|b_1\rangle + \sqrt{\lambda_2}|a_2\rangle|b_2\rangle + \sqrt{\lambda_3}|a_3\rangle|b_3\rangle$ be a Schmidt decomposition of $|\Psi\rangle$. Then

$$|\Psi\rangle \overset{LU}{\equiv} |\Psi'\rangle = \sqrt{\lambda_1}|0\rangle|0\rangle + \sqrt{\lambda_2}|1\rangle|1\rangle + \sqrt{\lambda_3}|2\rangle|2\rangle.$$
Unitary Evolution on Composite Systems

LU operators cannot generate entanglement!

\[(U \otimes V)|\alpha\rangle|\beta\rangle = U|\alpha\rangle V|\beta\rangle = |\alpha'\rangle|\beta'\rangle \quad \forall U, V, |\alpha\rangle, |\beta\rangle.\]

Any unitary that cannot be written as a tensor product is called a nonlocal unitary.

Example

The SWAP operator \(F\) is nonlocal on \(\mathcal{H} \otimes 2\):

\[F|\alpha\rangle|\beta\rangle = |\beta\rangle|\alpha\rangle \quad \forall |\alpha\rangle, |\beta\rangle.\]

Proof:
An important class of nonlocal gates on two-qubits is **controlled unitaries**. One type of controlled unitary has the form

\[ U^{AB} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes V, \]

where \( V \) is some unitary on system \( B \).

The unitary \( V \) is applied to \( B \) iff \( A \) is in state \( |1\rangle \):

\[ U|00\rangle = |00\rangle, \quad U|01\rangle = |01\rangle, \quad U|10\rangle = |1\rangle U|0\rangle, \quad U|11\rangle = |1\rangle U|1\rangle. \]

For this \( U \), we say system \( A \) is the “source” and system \( B \) is the “target.”

The operator \( \hat{U} = FU\hat{F} \) switches the roles of \( A \) and \( B \) so that \( B \) is the source and \( A \) is the target.
Unitary Evolution on Composite Systems

The controlled-not (CNOT) gate is one of the most useful operations in quantum computing:

\[ U_{CNOT} = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes \sigma_x. \]

It has the action

\[
\begin{align*}
U_{CNOT}|00\rangle &= |00\rangle, \\
U_{CNOT}|01\rangle &= |01\rangle, \\
U_{CNOT}|10\rangle &= |11\rangle, \\
U_{CNOT}|11\rangle &= |10\rangle.
\end{align*}
\]

\[ U_{CNOT} \] does not generate entanglement when acting on the computational product basis. However, it can generate entanglement when acting on superpositions:

\[ U_{CNOT}|+\rangle|0\rangle = \sqrt{1/2}(|00\rangle + |11\rangle). \]
Non-Entangling Gates

We have seen that every local unitary cannot generate entanglement. But is the converse true? Does every nonlocal unitary generate entanglement?

The answer is no. SWAP (which is nonlocal) cannot generate an entangled state from a product state since $F|\alpha\rangle|\beta\rangle = |\beta\rangle|\alpha\rangle$.

In the homework, you will prove that SWAP and LU are the only types of non-entangling gates.

However, SWAP becomes entangling when acting on subsystems of some larger system. For example, if $|\Psi^{AA'}\rangle$ is entangled, then swapping systems $A'$ and $B$ in the state $|\Psi^{AA'}\rangle|0\rangle^B$ generates $|\Psi^{AB}\rangle|0\rangle^{A'}$. So systems $AA'$ and $B$ are entangled even though they started in a product state.
The Bell Basis

For two-qubit systems, so far we have primarily been working in the computational product basis \( \{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \} \). However, another highly useful basis for \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) is called the **Bell basis**, and it consists of four orthonormal entangled states:

\[
|\Phi^+\rangle := |\Phi_{00}\rangle = \sqrt{1/2}(|00\rangle + |11\rangle)
\]
\[
|\Psi^+\rangle := |\Phi_{01}\rangle = \sqrt{1/2}(|01\rangle + |10\rangle)
\]
\[
|\Phi^-\rangle := |\Phi_{10}\rangle = \sqrt{1/2}(|00\rangle - |11\rangle)
\]
\[
|\Psi^-\rangle := |\Phi_{11}\rangle = \sqrt{1/2}(|01\rangle - |10\rangle).
\]
The Bell Basis

Notice that the Bell states are LU equivalent to one another.

\[ |\Phi_{b_1 b_2}\rangle = \sigma_{z}^{b_1} \sigma_{x}^{b_2} \otimes \mathbb{I} |\Phi_{00}\rangle \quad b_1, b_2 \in \{0, 1\}. \]

In \( |\Phi_{b_1 b_2}\rangle \), \( b_1 \) is called the “phase” bit and \( b_2 \) is called that “amplitude” bit (remember the ordering “PA”).

Example

Express the state \( |\Psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \) in the Bell basis.