Quantum Data Compression Pt. 1

PHYS 500 - Southern Illinois University

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Classical versus Quantum Information

In general, communication between two individuals involves the exchange of codewords such phonetic words, strings of 0/1's, hand signals, etc. The two individuals have a pre-established dictionary of what the different codewords mean, and this allows them to communicate.

All forms of communication that we experience share the same fundamental property that any two codewords describe states of a physical system can be perfectly distinguished.

In this course we will identify classical information as being one particular state of a physical system whose possible states are, in principle, perfectly distinguishable from one another.
Classical versus Quantum Information

On small length scales, however, the laws of physics are described by quantum mechanics, and one prominent feature of quantum mechanics is the existence of non-distinguishable physical states.

What does it mean for quantum states to be distinguishable?

Suppose that Bob has a quantum system with state space $\mathcal{H}^B$ and Alice prepares his system in either state $|\psi_0\rangle$ or $|\psi_1\rangle$. We say that Bob is able to perfectly distinguish these states if there exists a two-outcome POVM $\{\Pi_0, \Pi_1\}$ such that

$$0 = \langle \psi_0 | \Pi_1 | \psi_0 \rangle = \langle \psi_1 | \Pi_0 | \psi_1 \rangle.$$
Classical versus Quantum Information

A quick review on POVMS:
Classical versus Quantum Information

Bob is able to perfectly distinguish the states $|\psi_0\rangle$ and $|\psi_1\rangle$ if there exists a two-outcome POVM \{\Pi_0, \Pi_1\} such that

$$0 = \langle \psi_0 | \Pi_1 | \psi_0 \rangle = \langle \psi_1 | \Pi_0 | \psi_1 \rangle.$$

Physically this says that with zero probability will Bob obtain measurement outcome $i$ if Alice prepares state $|\psi_j\rangle$ whenever $i \neq j$.

**Claim:** The states $|\psi_0\rangle$ and $|\psi_1\rangle$ are perfectly distinguishable iff they are orthogonal.
In general, most pairs of states $|\psi_0\rangle$ and $|\psi_1\rangle$ in Hilbert space will not be orthogonal.

Yet, we would still like to think of quantum systems as containing some sort of information, even if it is information that is not distinguishable in the sense described above.

We thus define **quantum information** as simply being one particular state of a quantum system.

Note, these are *physical* definitions of information. They are more general than how we use the word “information” in everyday speech.
Classical Data Compression: Review

The Picture:

For i.i.d. random variables $X^n$, we say $\delta$-good compression is achievable at rate $\frac{m}{n}$ if there exists an encoder $f : \mathcal{X}^n \rightarrow \{0, 1\}^m$ and a decoder $g : \{0, 1\}^m \rightarrow \mathcal{X}^n$ such that

$$Pr\{X^n = g(f(X^n))\} > 1 - \delta.$$

Shannon’s Data Compression Theorem

For any $\delta > 0$ and $R > H(X)$, $\delta$-good compression can always be achieved at rate $R$. 
The von Neumann Entropy

Our goal is to generalize this task to the quantum setting.

In the quantum case, the random variable $X$ is replaced by an ensemble of states $\mathcal{E} = \{|\psi_x\rangle, p_x\}_{x \in X}$, which physically describes a quantum system that is prepared in state $|\psi_x\rangle$ with probability $p_x$.

The quantum entropy of this ensemble is a function of the ensemble average $\rho = \sum_x p_x |\psi_x\rangle \langle \psi_x|$, and it is defined as

$$S(\rho) = -\text{tr}[\rho \log \rho].$$

This quantity is called the \textbf{von Neumann entropy} of the density matrix $\rho$.

The von Neumann entropy plays a role analogous to the Shannon entropy in the task of quantum data compression.
Quantum Data Compression

The Picture: