Quantum Data Compression Pt. 2

PHYS 500 - Southern Illinois University

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Classical Data Compression: Review

The Picture:

For i.i.d. random variables $X^n$, we say $\delta$-good compression is achievable at rate $\frac{m}{n}$ if there exists an encoder $f : \mathcal{X}^n \to \{0, 1\}^m$ and a decoder $g : \{0, 1\}^m \to \mathcal{X}^n$ such that

$$\Pr\{X^n = g(f(X^n))\} > 1 - \delta.$$ 

Shannon’s Data Compression Theorem

For any $\delta > 0$ and $R > H(X)$, $\delta$-good compression can always be achieved at rate $R$. 

Quantum Data Compression

Our goal is to generalize this task to the quantum setting.

In the quantum case, the random variable $X$ is replaced by an ensemble of states $\mathcal{E} = \{ |\psi_x\rangle, p_x \}_{x \in \mathcal{X}}$, which physically describes a quantum system that is prepared in state $|\psi_x\rangle$ with probability $p_x$.

The Picture:
Quantum Data Compression

**Shannon’s Data Compression Theorem**

For any $\delta > 0$ and $R > H(X)$, $\delta$-good compression can always be achieved at rate $R$.

What we want is the following:

**Quantum Data Compression Theorem**

For any $\delta > 0$ and $R > S(\rho)$, $\delta$-good compression can always be achieved at rate $R$.

But what is $S(\rho)$ and $\delta$-good compression in the quantum case?
The von Neumann Entropy

The quantum entropy of this ensemble is a function of the ensemble average $\rho = \sum_x p_x |\psi_x\rangle\langle\psi_x|$, and it is defined as

$$S(\rho) = -\text{tr}[\rho \log \rho].$$

This quantity is called the **von Neumann entropy** of the density matrix $\rho$.

The von Neumann entropy plays a role analogous to the Shannon entropy in the task of quantum data compression.

Note that $S(\rho) = H(\{p_x\})$ if and only if the ensemble states $|\psi_x\rangle$ are orthogonal. To compute $S(\rho)$ in general you need its spectral decomposition.

Also note that $S(\rho^{\otimes n}) = nS(\rho)$. 
The von Neumann Entropy

Example

Suppose \( \rho = \frac{1}{4} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \). Compute \( S(\rho) \).
Fidelity of Two Pure States

We want to define some notion of how “close” two pure states are to one another.

Consider a $d$-dimensional system $S$. Suppose the system is prepared in some unknown pure state.

The experimenter (call her Esther) wants to test if the system is in some particular state $|\psi\rangle$. She performs the “$|\psi\rangle$-test”, which consists in making the projective measurement $\{P_0 = |\psi\rangle\langle\psi|, P_1 = \mathbb{I} - |\psi\rangle\langle\psi|\}$.

The system passes the “$|\psi\rangle$-test if and only if she obtains measurement outcome 0.
Fidelity of Two Pure States

What is the probability that some state $|\phi\rangle$ passes the "$|\psi\rangle$-test"? It is

$$\text{Prob}\{ |\phi\rangle \text{ passes} \} = \langle \phi | P_0 | \phi \rangle = |\langle \psi | \phi \rangle|^2.$$ 

The **fidelity** of two pure states $|\psi\rangle$ and $|\phi\rangle$ is defined as

$$F(|\psi\rangle, |\phi\rangle) := |\langle \psi | \phi \rangle|^2.$$ Operationally it measures the probability that $|\phi\rangle$ would pass the "$|\psi\rangle$-test"; or equivalently the probability that $|\psi\rangle$ would pass the "$|\phi\rangle$-test."

Notice that $F(|\psi\rangle, |\phi\rangle) = 1$ iff $|\psi\rangle = |\phi\rangle$ and $F(|\psi\rangle, |\phi\rangle) = 0$ iff $|\psi\rangle$ and $|\phi\rangle$ are orthogonal.

The fidelity of a pure state $|\psi\rangle$ with mixed state $\rho$ is defined as

$$F^2(|\psi\rangle, \rho) := \langle \psi | \rho | \psi \rangle.$$
Let $\mathcal{E} = \{|\psi_x\rangle, p_x\}$ be an ensemble of states with ensemble average $\rho = \sum_x p_x |\psi_x\rangle \langle \psi_x|$. For any $\delta > 0$ and $R > S(\rho)$, $\delta$-good compression can always be achieved at rate $R$.

The compression protocol is called Schumacher compression...