Quantum Communication over an Entanglement-Assisted Classical Channel

The Picture
Quantum Communication over an Entanglement-Assisted Classical Channel

The goal of teleportation is to simulate a quantum channel using shared entanglement and classical communication.

The ingredients:

1. A two-qubit maximally entangled state \( \sqrt{1/2}(|a_0 b_0\rangle + |a_1 b_1\rangle)^{AB} \), where \( \{|a_0\rangle, |a_1\rangle\} \) and \( \{|b_0\rangle, |b_1\rangle\} \) are any orthonormal bases for \( \mathbb{C}^2 \); such a state is called an entangled bit (ebit).

2. A one-qubit noiseless classical communication channel.

The resource trade-off of teleportation:

\[
[\text{ebit}] + 2[\text{CC}] \geq [\text{QQ}].
\]
Quantum Teleportation

The Protocol

Alice wants to send an arbitrary state $|\eta\rangle = \alpha|0\rangle + \beta|1\rangle$ to Bob. They share the maximally entangled state $|\Phi^+\rangle = \sqrt{1/2}(|00\rangle + |11\rangle)$.

The joint state can be written as

$$|\eta\rangle^{A'}|\Phi^+\rangle^{AB} = \frac{1}{\sqrt{2}}(\alpha|0\rangle + \beta|1\rangle)^{A'}(|00\rangle + |11\rangle)^{AB}$$

$$= \frac{1}{2}((|\Phi^+\rangle + |\Phi^+\rangle)\alpha|0\rangle + (|\Psi^+\rangle + |\Psi^-\rangle)\alpha|1\rangle$$

$$+ (|\Psi^+\rangle - |\Psi^-\rangle)\beta|0\rangle + (|\Phi^+\rangle - |\Phi^-\rangle)\beta|1\rangle)$$

$$= \frac{1}{2}(|\Phi^+\rangle^{A'}(\alpha|0\rangle + \beta|1\rangle) + |\Phi^-\rangle^{A'}(\alpha|0\rangle - \beta|1\rangle)$$

$$+ |\Psi^+\rangle^{A'}(\beta|0\rangle + \alpha|1\rangle) + |\Psi^-\rangle^{A'}(\beta|0\rangle - \alpha|1\rangle).$$
Quantum Teleportation

Alice measures systems $A'A$ in the Bell basis $\{|\Phi^+\rangle, |\Phi^-\rangle, |\Psi^+\rangle, |\Psi^-\rangle\}$.

<table>
<thead>
<tr>
<th>Alice’s Measurement Outcome</th>
<th>Bob’s Post-Measurement State</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\Phi^+\rangle =</td>
</tr>
<tr>
<td>$</td>
<td>\Psi^+\rangle =</td>
</tr>
<tr>
<td>$</td>
<td>\Phi^-\rangle =</td>
</tr>
<tr>
<td>$</td>
<td>\Psi^-\rangle =</td>
</tr>
</tbody>
</table>

When she projects onto $|\Phi_{b_0b_1}\rangle$, Alice communicates her classical outcome $(b_0, b_1)$. Bob then performs the local unitary $\sigma_{x}^{b_1} \sigma_{z}^{b_0}$.

With probability one, Alice transmits $|\eta\rangle$ to Bob.
Quantum Teleportation

This protocol works *coherently*. What this means is that system $A'$ is teleported to $B$ even when $A'$ is entangled with some other system $C$.

The Picture
Quantum Teleportation

To see that teleportation preserves coherence, consider that the teleportation protocol is a generalized measurement across $A'AB$ with Kraus operators

$$
M_{00}^{A'AB} = |\Phi_{00}\rangle\langle\Phi_{00}|^{A'}A \otimes \mathbb{I}^B, \\
M_{01}^{A'AB} = |\Phi_{01}\rangle\langle\Phi_{01}|^{A'}A \otimes \sigma_x^B, \\
M_{10}^{A'AB} = |\Phi_{10}\rangle\langle\Phi_{10}|^{A'}A \otimes \sigma_z^B, \\
M_{11}^{A'AB} = |\Phi_{11}\rangle\langle\Phi_{11}|^{A'}A \otimes (\sigma_x \sigma_z)^B.
$$

For an entangled $|\psi\rangle^{A'C} = \sum_k |\eta_k\rangle^{A'}|\tilde{\varphi}_k\rangle^C$ with $|\eta_k\rangle = \alpha_k|0\rangle + \beta_k|1\rangle$,

$$
|\psi\rangle^{A'C} |\Phi_{00}\rangle^{AB} \\
= \sum_k \frac{1}{2} (|\Phi_{00}\rangle^{A'}A(\alpha_k|0\rangle + \beta_k|1\rangle)^B + |\Phi_{10}\rangle^{A'}A(\alpha_k|0\rangle - \beta_k|1\rangle)^B \\
+ |\psi_{01}\rangle^{A'A}(\beta_k|0\rangle + \alpha_k|1\rangle)^B + |\psi_{11}\rangle^{A'A}(\beta_k|0\rangle - \alpha_k|1\rangle)^B)|\tilde{\varphi}_k\rangle^C.
$$
Quantum Teleportation

So performing the generalized measurement \( \{ M_{b_0 b_1}^{A'AB} \} \) \( b_0, b_1 \in \{0,1\} \) generates the post-measurement state

\[
\frac{1}{\sqrt{1/4}} M_{b_0 b_1}^{A'AB} |\psi\rangle^{A'C} |\Phi^+\rangle^{AB} = |\Phi_{b_0 b_1} \rangle^{A'A} |\psi\rangle^{BC}.
\]

If we average over the different outcomes, we obtain a noiseless quantum channel (CPTP map) \( \Lambda_{T_0, \Phi^+}^{A' \rightarrow B} \):

\[
X^{A'} \mapsto \Lambda_{T_0, \Phi^+}^{A' \rightarrow B}(X) = tr_{A'A}[ \sum_{b_0, b_1} M_{b_0 b_1}^{A'AB} (X^{A'} \otimes |\Phi^+\rangle \langle \Phi^+|^{AB}) M_{b_0 b_1}^{\dagger} ] = X^{B}.
\]

\((T_0, \Phi^+)\) refers to the standard teleportation protocol performed on \( |\Phi^+\rangle \).
Quantum Teleportation

Example

Generate any three-qubit entangled pure state using 2 ebits + CC.