The Framework of Quantum Mechanics

We now use the mathematical formalism covered in the last lecture to describe the theory of quantum mechanics. In the first section we outline four axioms that lie at the foundation of quantum mechanics. From these four axioms we derive in the second section an effective theory of quantum mechanics that applies to open quantum systems.

1 The Axioms of Quantum Mechanics

The basic framework of quantum mechanics consists of four axioms, and the subject of quantum mechanics involves studying what consequences follow from these axioms. This axiomatic approach is justified by repeated experimental verification of the theoretical predictions it makes.

1.1 The State Space Axiom

Every quantum system is represented by a Hilbert space \( \mathcal{H} \) called state space. Physical states of the system are in a one-to-one correspondence with rays in the Hilbert space.

Rays in a vector space are simply one-dimensional subspaces. The state space axiom therefore says that states of a quantum systems are identified with an entire one-dimensional subspace \( \{ \alpha |\psi\rangle : \alpha \in \mathbb{C} \} \), where \( |\psi\rangle \) is some unit vector. When performing calculations the convention is to represent physical states by unit vectors, and so henceforth we will assume that all kets are normalized unless otherwise stated. It is understood that multiplying this ket by an overall scalar does not change the physical state it represents. In particular, multiplying \( |\psi\rangle \) by an overall phase \( e^{i\phi} \) does not change any of the experimental outcomes predicted in quantum mechanics. This will be explained in greater detail when we discuss the Measurement Axiom.

There is another type of phase that does distinguish one state from another and which can lead to different experimental predictions. For a state \( |\psi\rangle \) decomposed in a linear combination

\[
|\psi\rangle = \alpha |0\rangle + \beta |1\rangle,
\]

a relative phase is a factor \( e^{i\phi} \) that is multiplied to just one of the kets but not both. For example, we would say that the vector

\[
|\psi'\rangle = \alpha |0\rangle + \beta e^{i\phi} |1\rangle
\]

differs from the vector \( |\psi\rangle \) by an relative phase, and the two represent different physical states of the system.

The State Space Axiom does not tell you the dimension of the Hilbert space associated with a given physical system. Nor does the State Space Axiom tell you which physical states of the system correspond to which rays in the Hilbert space. How both of these assignments are made depends on experimental observation. To understand this better, we need a way to describe observation and measurement in quantum mechanics. The Measurement Axiom will stipulate how this is done, but first we address of how quantum systems evolve in time.
1.2 The Unitary Evolution Axiom

Every closed quantum system evolves unitarily in time.

In more detail, let $\mathcal{H}$ be the state space for a closed quantum system, and consider any two moments in time, $t_0$ and $t_1$. The axiom says that there exists a unitary operator $U(t_0, t_1)$ acting on $\mathcal{H}$ such that if the system is in state $|\psi\rangle$ at time $t_0$, it will be in state $U(t_0, t_1)|\psi\rangle$ at time $t_1$. Since the evolution is unitary, $U(t_0, t_1)|\psi\rangle$ is still a unit vector. In addition, the entire process can be reversed by applying the unitary $U(t_0, t_1)^\dagger$.

Note that the axiom applies to closed quantum systems. It then becomes important to clarify the meaning of a closed system. In the language of physics, a system is closed if it is isolated, in the sense that it does not exchange energy or particles with any other system. A natural question is how open quantum systems evolve. We will address part of this question in Sect. ??.

1.3 The Measurement Axiom

Let $\mathcal{H}$ be the state space of some quantum system. A measurement on system $\mathcal{H}$ is represented by a hermitian operator $X \in \text{L}(\mathcal{H})$ called an observable, and conversely, every hermitian operator $X \in \text{L}(\mathcal{H})$ corresponds to a physical measurement on $\mathcal{H}$.

For an observable $X$ with spectral decomposition $X = \sum_{k=1}^n \lambda_k P_{\lambda_k}$ its eigenvalues $\lambda_k$ are the different values that can be measured when performing the measurement described by $X$. If $|\psi\rangle \in \mathcal{H}$ is the pre-measurement state of the system, then value $\lambda_k$ will be measured with probability

$$p(\lambda_k) = \langle \psi | P_{\lambda_k} | \psi \rangle. \quad (1)$$

When $\lambda_k$ is measured, the post-measurement state of the system is given by

$$\frac{1}{\sqrt{p(\lambda_k)}} P_{\lambda_k} |\psi\rangle. \quad (2)$$

There is quite a bit being asserted in the Measurement Axiom. The essential idea is that measurement in quantum mechanics involves a stochastic process. That is, one can only assign a probability distribution $\{p(\lambda_k)\}_{k=1}^n$ to the $n$ different outcomes of a quantum measurement. In general, it cannot be known in advance what the outcome of a quantum measurement will be with full certainty. The Measurement Axiom also shows why overall phases have no physical meaning in quantum mechanics. If we have two states $|\psi\rangle$ and $|\psi'\rangle = e^{i\theta}|\psi\rangle$, then

$$p(\lambda_k) = \langle \psi | P_{\lambda_k} | \psi \rangle = \langle \psi' | P_{\lambda_k} | \psi' \rangle.$$

In other words, all measurement outcomes are predicted to occur with the same probability whether we identify a physical state by $|\psi\rangle$ or by $e^{i\theta}|\psi\rangle$.

Equation (2) describes the state of the system after the measurement. Conditioned on outcome $\lambda_k$, the system undergoes the transformation

$$|\psi\rangle \rightarrow \frac{1}{\sqrt{p(\lambda_k)}} P_{\lambda_k} |\psi\rangle. \quad (3)$$

The factor of $\frac{1}{\sqrt{p(\lambda_k)}}$ on the post-measurement state is to ensure that it is normalized. This transformation is sometime referred to as “collapse of the wave function.” In this course we are not
States represented by eigenvectors of an observable are called eigenstates, and they have a special status in the measurement process. Let $|\lambda_k\rangle$ be any eigenvector in the eigenspace $V_{\lambda_k}$ of $X$. Then $P_{\lambda_k}|\lambda_k\rangle = \delta_{lk}|\lambda_k\rangle$, which means that the if $l \neq k$, there is zero probability of obtaining outcome $\lambda_l$ when the system is prepared in state $|\lambda_k\rangle$. The only possible outcome is $\lambda_k$, and the state $|\lambda_k\rangle$ remains unchanged in the measurement process. For this reason, eigenstates of an observable are referred to as stationary states.

Moreover, if an arbitrary state $|\psi\rangle$ is measured and outcome $\lambda_k$ is obtained, then quantum mechanics predicts that $\lambda_k$ will necessarily be obtained again if the same measurement is immediately performed a second time. The reason is that after the first measurement the system has collapsed into a stationary state $\frac{1}{\sqrt{p(\lambda_k)}} P_{\lambda_k} |\psi\rangle$. Typically, the system will begin undergoing some nontrivial unitary evolution as soon as the first measurement is complete. This will generally cause the system to leave the stationary state, and thus the outcome of the second measurement will retain uncertainty as time progresses between the two measurements.

We can now answer the questions of (i) how to decide the dimension of state space associated with some physical system, and (ii) how to assign the physical states of the system to rays in state space. For the first question, one could prepare multiple copies of the same physical system and carefully perform the same type of experimental observation on each of the copies. Due to the stochastic nature of quantum measurement, different outcomes will be observed among the different copies. The experimenter counts the total number of distinct outcomes, and this represents one estimate on the dimension of the quantum system. Indeed, by the measurement axiom, whatever observation the experimenter performs, it must be described by a hermitian operator. The eigenspaces of the operator provide a direct sum decomposition of state space with the number of eigenspaces corresponding to the number of different outcomes observed in the experiments. If all the eigenspaces are one-dimensional, then this number is precisely the dimension of state space. However, if there are degenerate eigenvalues, then this number will be strictly less than the dimension. In the latter case, the experimenter would need to perform “finer-grained” measurements described by observables not having degeneracy on this eigenspace. It is not a priori given what these finer-grained measurements would be if, in fact, there is degeneracy. Instead, one usually first has a theoretical model hypothesizing the Hilbert space structure of the system, and then this model is experimentally tested.

For the second question, experimental observation also places restrictions on the correspondence between physical states and kets in state space. We have discussed above how eigenvectors of an observable must be assigned to the stationary states of the measurement in order to be consistent with the Measurement Axiom. By the spectral decomposition of a given observable, it follows that in a $d$-dimensional Hilbert space, there exists $d$ stationary states for the observable whose corresponding kets form an orthonormal basis for state space. More generally, a physical state can be represented by a ket $|\psi\rangle$ if measurement outcome $\lambda_k$ is observed at frequency $p(\lambda_k) = \langle \psi | P_{\lambda_k} |\psi\rangle$ whenever the system is prepared in that state and the observable $X = \sum_{k=1}^{m} \lambda_k P_{\lambda_k}$ is performed.

For example, the Stern-Gerlach experiment revealed that the directional spin of an electron has two possible values. Thus, an electron spin system is described by a two-dimensional Hilbert space; i.e. $\mathcal{H} \cong \mathbb{C}^2$. We can represent the $\hat{z}$-direction “spin-up” eigenstate by ket $|0\rangle$ and the “spin-down” eigenstate by $|1\rangle$. Any other spin state of the electron is represented by a superposition of these states

$$|\psi\rangle = a|0\rangle + b|1\rangle,$$
where $|a|^2 + |b|^2 = 1$. Here, $|a|^2$ and $|b|^2$ represent the frequency of observing “spin-up” and “spin-down,” respectively, when the system is prepared in state $|\psi\rangle$ and spin is measured in the z-direction. From our discussion on the Stern-Gerlach experiment, an x-direction “spin-up” eigenstate measures “spin-up” and “spin-down” in the z-direction with equal probability. Thus if we let $|+\rangle$ denote the x-direction “spin-up” eigenstate, we must have $|+\rangle = a|0\rangle + b|1\rangle$ with $|a|^2 = |b|^2 = 1/2$. Up to an overall phase, $|+\rangle$ has the form $|+\rangle = \sqrt{1/2}(|0\rangle + e^{i\theta}|1\rangle)$. The x-direction “spin-down” eigenstate is orthogonal to $|+\rangle$, and thus has the form $|-\rangle = \sqrt{1/2}(|0\rangle - e^{i\theta}|1\rangle)$.

A final remark on the Measurement Axiom involves taking the expectation value of a quantum measurement. Suppose that $X = \sum_{k=1}^n \lambda_k P_{\lambda_k}$ is some quantum observable. By the Measurement Axiom, for pre-measurement state $|\psi\rangle$ the outcomes $\lambda_k$ are distributed according to the distribution $p(\lambda_k) = \langle \psi | P_{\lambda_k} | \psi \rangle$. This means that the expected value $\langle X \rangle$ of the measurement is given by

$$\langle X \rangle = \sum_{k=1}^n \lambda_k \langle \psi | P_{\lambda_k} | \psi \rangle = \langle \psi | \sum_{k=1}^n \lambda_k P_{\lambda_k} | \psi \rangle = \langle \psi | X | \psi \rangle.$$  

(4)

1.4 The Composite System Axiom

If $A$ and $B$ are two quantum systems with state spaces $\mathcal{H}^A$ and $\mathcal{H}^B$ respectively, the state space of the their combined physical system is the tensor product space $\mathcal{H}^{AB} := \mathcal{H}^A \otimes \mathcal{H}^B$. For more systems, $\mathcal{H}^{A_1, \mathcal{H}^{A_2}, \cdots, \mathcal{H}^{A_n}}$, the joint state space is the n-fold tensor product space $\mathcal{H}^{A_1 \otimes \mathcal{H}^{A_2} \otimes \cdots \otimes \mathcal{H}^{A_n}}$. Two quantum systems are often called a bipartite system while more than two systems are typically referred to as a multipartite system.

We already have good practice working with tensor product spaces in the previous lecture. Recall we emphasized that $\mathcal{H}^A \otimes \mathcal{H}^B$ is much larger than the set of tensor product states. As we will see, the difference between these two sets has dramatic physical consequences. A bipartite state $|\psi\rangle^{AB}$ is entangled if it is not a tensor product state; i.e. $|\psi\rangle^{AB} \neq |\alpha\rangle^A |\beta\rangle^B$. A non-entangled state is also called a product or separable state. In multipartite systems, a state $|\psi\rangle^{A_1A_2\cdots A_n}$ is entangled iff it is not an n-fold tensor product of states; i.e. $|\psi\rangle^{A_1A_2\cdots A_n} \neq |\alpha_1\rangle^{A_1} |\alpha_2\rangle^{A_2} \cdots |\alpha_n\rangle^{A_n}$. However, note that under this definition, states of the form $|\psi\rangle^{AB}|0\rangle^C$ are considered entangled whenever $|\psi\rangle^{AB}$ is entangled, despite the fact that system C is in a product state with $A$ and $B$. In multipartite systems, more interesting entanglement structures emerge than what is found in the bipartite case. For the time being we will only focus on bipartite systems and return to the multipartite scenario later in the course.

Let us consider how to determine whether a given $|\psi\rangle \in \mathcal{H}^{AB}$ is entangled. The following criterion for entanglement follows almost trivially from the isomorphism $\mathcal{H}^{AB} \cong L(\mathcal{H}^B \otimes \mathcal{H}^A)$ established in the last lecture.

**Proposition 1.** A bipartite state $|\psi\rangle$ is a product state iff its operator representation $M_{|\psi\rangle}$ is rank one.

**Proof.** If $|\psi\rangle = |\alpha\rangle |\beta\rangle$, then $M_{|\psi\rangle} = |\alpha\rangle \langle \beta|$, which is a rank one operator. Conversely, if $M_{|\psi\rangle} = |\alpha\rangle \langle \beta|$ is rank one, then $|\psi\rangle = M_{|\psi\rangle} \otimes I |\alpha\rangle |\beta\rangle$. \qed

From elementary matrix theory, a matrix has rank $r$ iff all of its $(r + 1)$-minors vanish, and there exists at least one nonvanishing $r$-minor. In $2 \times 2$ matrices, this reduces to the condition that a nonzero matrix has vanishing determinant. Thus, as an example consider an arbitrary two-qubit state $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$. The operator $M_{|\psi\rangle}$ has matrix representation
$M_{|\psi\rangle} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, whose rank is one iff its determinant is zero. That is, the necessary and sufficient condition for $|\psi\rangle$ to be separable is that $ad - bc = 0$. 