Quantum Teleportation Pt. 2

PHYS 500 - Southern Illinois University

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Teleporting Higher-Dimensional Systems

The teleportation protocol allows us to establish the following resource trade-off

\[ ebit + 2[\overrightarrow{CC}] \geq [\overrightarrow{QQ}] \]

But what if Alice and Bob want to simulate more than a single qubit channel?

Since coherence works coherently, a state of arbitrary dimension can be teleported from Alice to Bob:

\[ n[ebit] + 2n[\overrightarrow{CC}] \geq n[\overrightarrow{QQ}] \]

\[ n \text{ ebits} + 2n \text{ cbit channel} \geq n \text{ qubit channel.} \]
Teleporting Higher-Dimensional Systems

In more detail, suppose that $|\eta\rangle^{A'} = \sum_{i=1}^{d} c_i |i\rangle$ is an arbitrary $d$-dimensional state in $\mathbb{C}^d$.

The idea is to think of $\mathbb{C}^d$ as the tensor product space of $\lceil \log d \rceil$ qubits. Since $d \leq 2^{\lceil \log d \rceil}$, $|\eta\rangle^{A'} \in \mathbb{C}^d \subset (\mathbb{C}^2)^\otimes \lceil \log d \rceil$.

Relabel the basis vectors (this is nothing but a relabeling!):

$$\left\{ |i\rangle^{A'} : i = 1, \cdots, d \right\} \rightarrow \left\{ |b_1\rangle^{A'_1} |b_2\rangle^{A'_2} \cdots |b_{\lceil \log d \rceil}\rangle^{A'_{\lceil \log d \rceil}} : b_i \in \{0, 1\} \right\}.$$
Teleporting Higher-Dimensional Systems

Then

$$|\eta\rangle^{A'} = \sum_{b_i \in \{0,1\}} c_b |b_1\rangle^{A_1'} |b_2\rangle^{A_2'} \cdots |b_{\lceil \log d \rceil}\rangle^{A'_{\lceil \log d \rceil}}$$

where $\vec{b} = (b_1, b_2, \cdots, b_{\lceil \log d \rceil})$.

To teleport $|\eta\rangle$, Alice and Bob need $\lceil \log d \rceil$ ebits:

$$|\Phi^+\rangle^{A_1 B_1} |\Phi^+\rangle^{A_2 B_2} \cdots |\Phi^+\rangle^{A_{\lceil \log d \rceil} B_{\lceil \log d \rceil}}.$$ 

Then with $2\lceil \log d \rceil$ bits of classical communication, Alice can teleport systems $A_1' A_2' \cdots A'_{\lceil \log d \rceil}$ to Bob.

After performing error correction on all the systems $B_i$, Bob obtains $|\eta\rangle$. 
Suppose that Alice and Bob share a bipartite state $\rho^{AB}$ that is not maximally entangled; i.e. $\rho^{AB} \neq |\Phi^+\rangle\langle\Phi^+|^{AB}$.

What happens if the use $\rho^{AB}$ for teleportation?

For a concrete example, consider the pure state $|\Phi_\kappa\rangle = \cos \kappa |00\rangle + \sin \kappa |11\rangle$.

They perform the standard teleportation protocol $\mathcal{T}_0$ that consists of:

1. Alice performs a Bell measurement $\{ |\Phi_{b_0b_1}\rangle \}_{b_0,b_1\in\{0,1\}}$ and communicates the result to Bob.
2. Bob performs the local unitary $\sigma^z_{b_0} \sigma^x_{b_1}$. 
Teleportation without Maximal Entanglement

For input state $|\eta\rangle^{A'} = \alpha|0\rangle + \beta|1\rangle$, they share

$|\eta\rangle^{A'}|\Phi_\kappa\rangle^{AB}$

$= (\alpha|0\rangle + \beta|1\rangle)^{A'} (\cos \kappa|00\rangle + \sin \kappa|11\rangle)^{AB}$

$= \frac{1}{\sqrt{2}} \left( (|\Phi^+\rangle + |\Phi^−\rangle)\alpha \cos \kappa|0\rangle + (|\Psi^+\rangle + |\Psi^−\rangle)\alpha \sin \kappa|1\rangleight.$

$\left. + (|\Psi^+\rangle − |\Psi^−\rangle)\beta \cos \kappa|0\rangle + (|\Phi^+\rangle − |\Phi^−\rangle)\beta \sin \kappa|1\rangle \right)$

$= \frac{1}{\sqrt{2}} [|\Phi^+\rangle^{A'A'} (\alpha \cos \kappa|0\rangle + \beta \sin \kappa|1\rangle)^B + |\Phi^−\rangle^{A'A'} (\alpha \cos \kappa|0\rangle − \beta \sin \kappa|1\rangle)^B$

$+ |\psi^+\rangle^{A'A'} (\beta \cos \kappa|0\rangle + \alpha \sin \kappa|1\rangle)^B + |\psi^−\rangle^{A'A'} (−\beta \cos \kappa|0\rangle + \alpha \sin \kappa|1\rangle)^B].$
Teleportation without Maximal Entanglement

Alice measures in the Bell basis:

\[ Prob\{|\Phi^+\rangle\} = Prob\{|\Phi^-\rangle\} = \frac{1}{2}(|\alpha|^2 \cos^2 \kappa + |\beta|^2 \sin^2 \kappa) =: p_\Phi, \]

\[ Prob\{|\Psi^+\rangle\} = Prob\{|\Psi^-\rangle\} = \frac{1}{2}(|\beta|^2 \cos^2 \kappa + |\alpha|^2 \sin^2 \kappa) =: p_\Psi. \]

<table>
<thead>
<tr>
<th>Alice’s Outcome</th>
<th>Bob’s Post-Measurement</th>
<th>After Bob’s Error Correction</th>
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How well does this protocol work?
Fidelity of Two Pure States

We want to define some notion of how “close” two pure states are to one another. This will allow us to assess how “well” the teleportation with $|\Phi_\kappa\rangle$ works.

Consider a $d$-dimensional system $S$. Suppose the system is prepared in some unknown pure state.

The experimenter (call her Esther) wants to test if the system is in some particular state $|\psi\rangle$. She performs the “$|\psi\rangle$-test”, which consists in making the projective measurement $\{P_0 = |\psi\rangle\langle\psi|, P_1 = \mathbb{I} - |\psi\rangle\langle\psi|\}$.

The system passes the “$|\psi\rangle$-test if and only if she obtains measurement outcome 0.
Fidelity of Two Pure States

What is the probability that some state $|\phi\rangle$ passes the “$|\psi\rangle$-test”? It is

$$\text{Prob}\{ |\phi\rangle \text{ passes} \} = \langle \phi | P_0 | \phi \rangle = |\langle \psi | \phi \rangle|^2.$$ 

**Definition**

The **fidelity** of two pure states $|\psi\rangle$ and $|\phi\rangle$ is defined as

$$F(|\psi\rangle, |\phi\rangle) := |\langle \psi | \phi \rangle|^2.$$  Operationally it measures the probability that $|\phi\rangle$ would pass the “$|\psi\rangle$-test”; or equivalently the probability that $|\psi\rangle$ would pass the “$|\phi\rangle$-test.”

Notice that $F(|\psi\rangle, |\phi\rangle) = 1$ iff $|\psi\rangle = |\phi\rangle$ and $F(|\psi\rangle, |\phi\rangle) = 0$ iff $|\psi\rangle$ and $|\phi\rangle$ are orthogonal.
Teleportation Fidelity

Suppose that Alice and Bob perform the standard teleportation protocol $\mathcal{T}_0$ using some entangled state $\rho^{AB}$.

Let $\{ |\eta_{00}\rangle, |\eta_{01}\rangle, |\eta_{10}\rangle, |\eta_{11}\rangle \}$ denote the four possible final states that Bob obtains corresponding to Alice’s measurement outcomes $\{00, 01, 10, 11\}$ when she attempts to send the state $|\eta\rangle$.

The average fidelity of Bob’s final state with the input state $|\eta\rangle$ is

$$\frac{1}{4} \sum_{b_0, b_1 = 0} p_{b_0b_1} |\langle \eta | \eta_{b_0b_1} \rangle|^2,$$

where $p_{b_0b_1}$ is the probability that Alice obtains the measurement outcome $(b_0, b_1)$. 
Teleportation Fidelity

Let us suppose that Alice chooses $|\eta\rangle = \alpha|0\rangle + \beta|1\rangle$ randomly among all possible qubit states. What is the fidelity of teleportation when averaging over all possible states $|\eta\rangle$?

To make this calculation, recall that every qubit state can be identified with a point on the Bloch sphere:

$$|\eta\rangle = |\hat{n}\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle,$$

where $\alpha = \cos(\theta/2)$ and $\beta = e^{i\phi}\sin(\theta/2)$.

Randomly choosing a state $|\hat{n}\rangle$ is equivalent to randomly choosing a point on the Bloch sphere.
Teleportation Fidelity

The uniform probability density function on the sphere is \( \frac{1}{4\pi} \sin \theta \).
The probability of randomly choosing a point within some region \( R \) on the unit sphere is

\[
\frac{\text{Area of } R}{\text{Area of sphere}} = \frac{\int_R \sin \theta d\theta d\phi}{4\pi}.
\]

Definition

The **teleportation fidelity** for protocol \( T_0 \) using the shared state \( \rho^{AB} \) is

\[
f_{T_0}(\rho) := \int d\hat{n} \sum_{b_0,b_1=0}^{1} p_{b_0 b_1} |\langle \hat{n} | \eta_{b_0 b_1} \rangle|^2 \quad \left( d\hat{n} = \frac{\sin \theta}{4\pi} d\theta d\phi \right)
\]

\[
= \int \int \frac{d\theta d\phi}{4\pi} \sin \theta \sum_{b_0,b_1=0}^{1} p_{b_0 b_1} |\langle \hat{n} | \eta_{b_0 b_1} \rangle|^2.
\]
Teleportation Fidelity

Let us compute $f_{T_0}(\Phi_\kappa)$, the teleportation fidelity using $|\Phi_\kappa\rangle$ and the standard protocol $T_0$.

First suppose that Alice sends $|\eta\rangle = \alpha|0\rangle + \beta|1\rangle$. Bob’s final states:

$$\begin{align*}
\sum_{b_0,b_1=0}^1 p_{b_0 b_1} |\langle \eta | \eta_{b_0 b_1} \rangle|^2 &= |\alpha|^2 \cos \kappa + |\beta|^2 \sin \kappa|^2 + |\alpha|^2 \sin \kappa + |\beta|^2 \cos \kappa|^2.
\end{align*}$$
Make the substitutions $\alpha = \cos(\theta/2)$ and $\beta = e^{i\phi} \sin(\theta/2)$ and compute

$$\int \int \frac{d\theta d\phi}{4\pi} \sin \theta \sum_{b_0,b_1=0}^{1} p_{b_0 b_1} |\langle \hat{n} | \eta_{b_0 b_1} \rangle|^2.$$ 

$$f_{T_0}(\Phi_\kappa) = \frac{2}{3}(1 + \cos \kappa \sin \kappa).$$
Figure: The teleportation fidelity as a function of $\kappa$ when performing the standard teleportation protocol $\mathcal{T}_0$ using the shared state $\cos \kappa |00\rangle + \sin \kappa |11\rangle$. 