Group Theory pt 2

PHYS 500 - Southern Illinois University

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SO(3)

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Every three-dimensional rotation can be defined by Euler angles $(\phi, \theta, \psi)$:

1. Rotation by angle $\phi$ about the $z$-axis;
2. Rotation by angle $\theta$ about the line of nodes;
3. Rotation by angle $\psi$ about the $z'$-axis.

The rotation matrix is given by $R(\phi, \theta, \psi) = R_z(\phi)R_y(\theta)R_z(\psi)$.

For a rotation of angle $\Phi$ about the axis $\hat{n} = (\theta, \phi)$, the rotation matrix is $R_{\hat{n}}(\Phi) = R(\phi, \theta, \zeta)R_z(\Phi)R_{\phi, \theta, \zeta}$ for arbitrary $\zeta$. 
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For a rotation of angle $\Phi$ about the axis $\mathbf{n} = (\theta, \phi)$, the rotation matrix is

$$R_{\mathbf{n}}(\Phi) = R(\phi, \theta, \zeta)R_z(\Phi)R^{-1}(\phi, \theta, \zeta)$$

for arbitrary $\zeta$. 
Cosets

Definition

If $\mathcal{G}$ is a group, then a subset $\mathcal{H} \subseteq \mathcal{G}$ is a subgroup if it forms a group under the group operations of $\mathcal{G}$.
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**Definition**
Let \( H \) be a subgroup of \( G \) and let \( g \) be any element of \( G \). The set

\[ gH = \{ gh : h \in H \} \]

is called a **left coset** of \( H \) in \( G \). The set

\[ Hg = \{ hg : h \in H \} \]

is called a **right coset** of \( H \) in \( G \).
Property of Cosets

If $\mathcal{H} \subset G$ is a subgroup, then for every $a, b \in G$, either:

1. $a\mathcal{H} = b\mathcal{H}$, or
2. $a\mathcal{H} \cap b\mathcal{H} = \emptyset$ and $|a\mathcal{H}| = |b\mathcal{H}|$. 

Proof.
Lagrange's Theorem

Let $G$ be a group and $H$ any subgroup. Then $|G| = n \in \mathbb{Z}$. 

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$$\frac{|\mathcal{G}|}{|\mathcal{H}|} = n \in \mathbb{Z}.$$
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Let \( G \) be a group and \( H \) any subgroup. Then

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\frac{|G|}{|H|} = n \in \mathbb{Z}.
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Conjugacy Classes

Definition
For a group $G$, two elements $a, b \in G$ are called conjugate if there exists a $g \in G$ such that

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Examples: $D_3$;
$SO(3)$ - the conjugacy classes are characterized by angle $\Phi \in [0, \pi]$. 
Conjugacy Classes

An element \( a \in G \) is self-conjugate if \( gag^{-1} = a \) for all \( g \in G \).
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An element $a \in G$ is self-conjugate if $gag^{-1} = a$ for all $g \in G$.

**Definition**

The center of a group $G$ is the set

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**Definition**

Two subgroups $\mathcal{H}$ and $\mathcal{H}'$ are **conjugate** if there exists some $g \in G$ such that $[\mathcal{H} = g\mathcal{H}'g^{-1}]$. An **invariant subgroup** is any subgroup $\mathcal{H} \subset G$ such that $g\mathcal{H}g^{-1} = \mathcal{H}$ for all $g \in G$. 
The Quotient Group

If $\mathcal{H}$ is an invariant subgroup, then $g\mathcal{H} = \mathcal{H}g$ for any $g \in G$. 
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$\mathcal{G} \setminus \mathcal{H}$

For an invariant subgroup $\mathcal{H}$, the cosets of $\mathcal{H}$ form a group $\mathcal{G} \setminus \mathcal{H}$.

Group operation: $(g_1 \mathcal{H})(g_1 \mathcal{H}) = g_1 g_2 \mathcal{H}$. 
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Example

$\mathcal{H} = \{e, d, d^{-1}\} \subset D_3$. Note that $\mathcal{H}$ is isomorphic to $\mathbb{Z}_3$, and $D_3 \setminus \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_2$. 
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Definition

Let $\mathcal{H}$ be a subgroup of $\mathcal{G}$. A mapping $f : \mathcal{G} \rightarrow \mathcal{H}$ is called a **homomorphism** if it preserves group multiplication. The elements $g \in \mathcal{G}$ mapped to the identity is the kernel of $f$, denoted by $\ker f$. 
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For an invariant subgroup $\mathcal{H}$, the mapping $f : \mathcal{G} \rightarrow \mathcal{G} \setminus \mathcal{H}$ is a homomorphism.