Hilbert Spaces: Infinite-Dimensional Vector Spaces

PHYS 500 - Southern Illinois University

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Infinite-Dimensional Vector Spaces: $\ell^2$

Infinite dimensional vector spaces are vector spaces that cannot be spanned by a finite number of elements.

Example ($\ell^2$)

A prime example of an infinite-dimensional vector space is $\ell^2$. This is the subset of infinite-length sequences:

$$\ell^2 := \left\{ x = (x_1, x_2, \cdots) \in C^\infty : \sum_{k=1}^{\infty} |x_k|^2 < \infty \right\}.$$
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Vector addition in $\ell^2$ is defined component-wise:

$$x + y = (x_1, x_2, \cdots) + (y_1, y_2, \cdots) := (x_1 + y_1, x_2 + y_2, \cdots).$$
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Properties of $\ell^2$

- $\ell^2$ has an inner product defined as

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k^* y_k.$$ 

- The norm of a vector $x \in \ell^2$ is given by $\|X\| = \sqrt{\langle x, x \rangle}$. 
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Note that $(x, y)$ is finite for $x, y \in \ell^2$ since

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\sum_{k=1}^{\infty} x_k^* y_k \leq \frac{1}{2} \sum_{k=1}^{\infty} (|x_k|^2 + |y_k|^2) < \infty.
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Properties of $l^2$

- $l^2$ is a **separable** vector space. Being separable means that it has a countable basis. The basis for $l^2$ is given by

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e_1 = (1, 0, 0, \cdots), \quad e_2 = (0, 1, 0, \cdots), \quad \cdots, \quad e_n = (0, \cdots, 1, 0, \cdots), \cdots$$
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Proof.
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Let $\mathcal{H}$ be a Hilbert space. A set of vectors $\{\phi_1, \phi_2, \cdots \}$ with $\phi_k \in \mathcal{H}$ is said to be an **orthonormal system** if $(\phi_i, \phi_j) = \delta_{ij} = 0$. 

Note

An orthonormal set of vectors $\{\phi_1, \phi_2, \cdots \}$ being "complete" is different than a vector space being "complete".
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Complete sets of vectors

Theorem

Let \( \{ \phi_1, \phi_2, \cdots \} \) be an orthonormal set for a Hilbert space \( \mathcal{H} \). The following statements are equivalent:

1. The set \( \{ \phi_1, \phi_2, \cdots \} \) is complete.
2. Every vector \( x \in \mathcal{H} \) can be expressed as \( x = \sum_{k=1}^{\infty} (\phi_k, x) \phi_k \).
3. Every vector \( x \in \mathcal{H} \) satisfies \( \|x\|^2 = \sum_{k=1}^{\infty} |(\phi_k, x)|^2 \).
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