One-Forms and Cotangent Space

PHYS 500 - Southern Illinois University

September 6, 2016
Recall

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It is easy to see that \( T_p(\mathcal{M}) \) is a vector space:

Tangent vector for \( \gamma(\lambda) = (x^1(\lambda), \cdots, x^n(\lambda)) \) at \( p \):
\[
\mathbf{v} = \frac{dx^k}{d\lambda} \frac{\partial}{\partial x^k} \bigg|_p
\]

Tangent vector for \( \hat{\gamma}(\lambda) = (\hat{x}^1(\lambda), \cdots, \hat{x}^n(\lambda)) \) at \( p \):
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\hat{\mathbf{v}} = \frac{d\hat{x}^k}{d\lambda} \frac{\partial}{\partial x^k} \bigg|_p
\]

Tangent vector for curve \( a\gamma(\lambda) + b\hat{\gamma}(\lambda) \) at \( p \):
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\left( a \frac{dx^k}{d\lambda} + b \frac{d\hat{x}^k}{d\lambda} \right) \frac{\partial}{\partial x^k} \bigg|_p
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For local coordinates \( \{x^k\}_k \) on a neighborhood of \( p \), the operators \( \{ \frac{\partial}{\partial x^k} \}_k \) provide a basis for \( T_p(\mathcal{M}) \).
One-Forms

Definition

To any complex vector space $V$ is associated a dual space $V^*$ which is the set of all complex-valued functions (functionals) acting on $V$. If $\tilde{\omega} \in V^*$ then

$$\tilde{\omega}(v) \in \mathbb{C} \quad \forall \; v \in V.$$
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Definition
Addition and scalar multiplication of elements $\tilde{\omega}, \tilde{\tau} \in V^*$ can be defined by $(a\tilde{\omega} + b\tilde{\tau})(v) = a\tilde{\omega}(v) + b\tilde{\tau}(v)$. Thus the set $V^*$ forms a vector space, and the elements $\tilde{\omega} \in V^*$ are called one-forms.
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Example
$\mathbb{C}^n$: 

One-Forms

Dual Basis

If the set of vectors \( \{e_k\} \) forms an basis for \( V \), then the set of functionals \( \{\tilde{\omega}^k\} \) satisfying

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\tilde{\omega}^j(e_k) = \delta^j_k
\]

is called the dual basis for \( V^* \).
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Notice that for an arbitrary vector \( \mathbf{v} = v^k e_k \in V \),

\[
\tilde{\omega}^j(\mathbf{v}) = \tilde{\omega}^j \left( v^k e_k \right) = v^k \tilde{\omega}^j(e_k) = v^k \delta^j_k = v^j,
\]

which is the \( j^{th} \) component of \( \mathbf{v} \).
Then if $\tilde{\tau}$ is an arbitrary one-form, we have

$$\tilde{\tau}(v) = v^k \tilde{\tau}(e_k) = \tilde{\omega}^k(v) \tilde{\tau}(e_k) = \tau_k \tilde{\omega}^k(v),$$

where $\tau_k = \tilde{\tau}(e_k)$. 
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Since this holds for an arbitrary $v$, we have the identity

$$\tilde{\tau} = \tau_k \tilde{\omega}^k.$$

The numbers $\tau_k$ are called the components of $\tilde{\tau}$ in the basis $\tilde{\omega}^k$. 
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Notation:

<table>
<thead>
<tr>
<th>$v = v^k e_k \in V$</th>
<th>Components</th>
<th>Basis Vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{\tau} = \tau_k \tilde{\omega}^k \in V^*$</td>
<td>$v^k$</td>
<td>$\tau_k$</td>
</tr>
</tbody>
</table>
The Cotangent Space

Definition

For a manifold \( \mathcal{M} \) and point \( p \in \mathcal{M} \), the dual space \( T^*_p(\mathcal{M}) \) to the tangent space \( T_p(\mathcal{M}) \) is called the \textbf{cotangent space}. The manifold \( \mathcal{M} \) together with the collection of cotangent spaces at each point in the manifold forms a product manifold called the \textbf{cotangent bundle} \( T^*\mathcal{M} \).
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For local coordinates $\{x^k\}_k$ of a neighborhood of $p$, the corresponding coordinate basis for $T_p(\mathcal{M})$ is $\{e_k = \frac{\partial}{\partial x^k}\}_k$. The dual basis of $\{\frac{\partial}{\partial x^k}\}_k$ for $T^*(\mathcal{M})$ is denoted by $\{dx^k\}_k$:

$$\tilde{dx}^j \left(\frac{\partial}{\partial x^k}\right)\bigg|_p = \delta^j_k.$$
The Cotangent Space

An arbitrary one-form $\tilde{\tau} \in T_p^*(\mathcal{M})$ can be expressed as

$$\tilde{\tau} = \tilde{\tau} \left( \frac{\partial}{\partial x^k} \right) \bigg|_p \tilde{dx}^k = \tau^k dx^k.$$ 

The $\tau_k = \tilde{\tau} \left( \frac{\partial}{\partial x^k} \right) |_p$ are the components of $\tilde{\tau}$ w.r.t. the local coordinate $\{x^k\}_k$ on $\mathcal{M}$. 
The Cotangent Space

An arbitrary one-form $\tilde{\tau} \in T^*_p(M)$ can be expressed as

$$\tilde{\tau} = \tilde{\tau} \left( \frac{\partial}{\partial x^k} \right) \bigg|_p \tilde{dx}^k = \tau_k \tilde{dx}^k.$$ 

The $\tau_k = \tilde{\tau} \left( \frac{\partial}{\partial x^k} \right) \big|_p$ are the components of $\tilde{\tau}$ w.r.t. the local coordinate $\{x^k\}_k$ on $M$.

How do the components of $\tilde{\tau}$ change under a change of local coordinates? Let $\{y^k\}$ be another set of components so that $\left\{ \frac{\partial}{\partial y^k} \right\}_k$ is another basis for $T_p(M)$ and $\{\tilde{dy}^k\}_k$ is another basis for $T^*_p(M)$. 


The Cotangent Space

Expand $\tilde{dx}^j$ in the $\tilde{dy}^k$ basis:

$$\tilde{dx}^j = \alpha_k \tilde{dy}^k.$$
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Now act on $\frac{\partial}{\partial y^l} = \frac{\partial x^k}{\partial y^l} \frac{\partial}{\partial x^k}$ to obtain:

$$\tilde{dx}^j = \frac{\partial x^j}{\partial y^k} \tilde{dy}^k.$$
The Cotangent Space

**Transformation law for one-form components**

If we represent $\tilde{\tau}$ in the two different basis $\tilde{dx}^j$ and $\tilde{dy}^k$, i.e.

$$\tilde{\tau} = \zeta_k \tilde{dx}^k = \eta_k \tilde{dy}^k,$$

then the components in these two bases are related by

$$\eta_k = \zeta_j \frac{\partial x^j}{\partial y^k}, \quad \zeta_k = \eta_j \frac{\partial y^j}{\partial x^k}.$$

One-form components transform opposite to tangent vectors components!

**Proof**
Differentials

Definition

For any smooth function $f$ defined on the manifold, its gradient defines a one-form for each cotangent space $T^*_p(M)$ on the manifold. For a vector $\frac{d}{d\lambda} = v^k \frac{\partial}{\partial x^k} \in T_p(M)$,

$$\tilde{df} \left( \frac{d}{d\lambda} \right) |_p = v^k \frac{\partial f}{\partial x^k} |_p.$$

$$= \frac{d}{d\lambda} (f) |_p.$$

The one-form $df = \tilde{df}$ is called the **differential** of $f$. 
Differentials

Definition

For any smooth function $f$ defined on the manifold, its gradient defines a one-form for each cotangent space $T_p^*(\mathcal{M})$ on the manifold. For a vector

$$\frac{d}{d\lambda} = v^k \frac{\partial}{\partial x^k} \in T_p(\mathcal{M}),$$


$$\widetilde{df} \left( \frac{d}{d\lambda} \right) |_p : = v^k \frac{\partial f}{\partial x^k} |_p.$$

$$= \frac{d}{d\lambda} (f)|_p.$$

The one-form $df = \widetilde{df}$ is called the **differential** of $f$.

The basis one forms $d\!x^k$ are the differentials of the coordinate functions $x^k(p)$. 

Duality:

\[
\frac{d}{d\lambda} \in T_p(\mathcal{M}) : T_p^*(\mathcal{M}) \to \mathbb{R}
\]

\[
df \in T_p^*(\mathcal{M}) : T_p(\mathcal{M}) \to \mathbb{R}
\]

\[
\left. \frac{d}{d\lambda} (df) \right|_p = df \left( \frac{d}{d\lambda} \right) \bigg|_p \in \mathbb{R}.
\]