Tensor Product Spaces

Definition

Let $U$ and $V$ be two vector spaces with arbitrary bases $\{u_j\}_{j=1}^{d_U}$ and $\{v_k\}_{k=1}^{d_V}$ respectively. Their tensor product $W = U \otimes V$ is the vector space with basis elements $\{u_j \otimes v_k\}_{j,k=1}^{d_U,d_V}$. The elements of $W$ are linear combinations of the basis vectors

$$w = \alpha^{j,k} u_j \otimes v_k$$

and are called tensors. Tensor addition satisfies

$$\alpha(u_j \otimes v_k) + \beta(u_j \otimes v_l) = u_j \otimes (\alpha v_k + \beta v_l)$$

$$\alpha(u_j \otimes v_k) + \beta(u_l \otimes v_k) = (\alpha u_j + \beta u_l) \otimes v_k.$$ 

Example

Quantum mechanics; $\mathbb{C}^n$. 
Tensor Products on the Manifold

For a manifold $\mathcal{M}$ and point $p \in \mathcal{M}$, we introduce the tensor product space

$$T_p^{(M,N)} := \bigotimes^{M} T_p(\mathcal{M}) \bigotimes^{N} T_p^*(\mathcal{M}),$$

which is $M$ copies of $T_p(\mathcal{M})$ and $N$ copies of $T_p^*(\mathcal{M})$. Elements of $T_p^{(M,N)}$ are called $\binom{M}{N}$ tensors.
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For local coordinates $\{x^k\}$, basis vectors of $T_p^{(M,N)}$ take the form

$$e_{i_1,\ldots,i_M}^{j_1,\ldots,j_N} = \frac{\partial}{\partial x^{i_1}} \bigotimes \cdots \bigotimes \frac{\partial}{\partial x^{i_M}} \bigotimes dx^{j_1} \bigotimes \cdots \bigotimes dx^{j_N}.$$
Tensor Products on the Manifold

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$$T_p^{(M,N)} := \bigotimes_{i=1}^{M} T_p(M) \bigotimes_{j=1}^{N} T_p^*(M),$$

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For local coordinates $\{x^k\}$, basis vectors of $T_p^{(M,N)}$ take the form

$$e_{i_1}^{j_1}, \cdots, e_{i_M}^{j_N} = \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_M}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_N}.$$

An arbitrary $\binom{M}{N}$ tensor $T$ takes the form $T = t_{j_1, \cdots, j_N}^{i_1, \cdots, i_M} e_{i_1}^{j_1}, \cdots, e_{i_M}^{j_N}$. 
Tensor Products on the Manifold

Contraction of indices

An \((M \choose N)\) tensor is a functional on the space \(T^p_{(N,M)}\).

Let \(T = t^i_{j_1, \ldots, j_N} e^j_{i_1, \ldots, i_M}\) be an \((M \choose N)\) tensor.

Let \(S = s^j_{i_1, \ldots, i_M} e^i_{j_1, \ldots, j_N}\) be an \((N \choose M)\) tensor.

Then: \(T(S) = S(T) = t^i_{j_1, \ldots, j_N} s^j_{i_1, \ldots, i_M}\).
Tensor Products on the Manifold

Contraction of indices

An \(\binom{M}{N}\) tensor is a functional on the space \(T_p^{(N,M)}\).

Let \(T = t_{j_1,\ldots,j_N}^{i_1,\ldots,i_M} e_{i_1,\ldots,i_M} e^{j_1,\ldots,j_N}\) be an \(\binom{M}{N}\) tensor.

Let \(S = s_{i_1,\ldots,i_M}^{j_1,\ldots,j_N} e^{i_1,\ldots,i_M} e_{j_1,\ldots,j_N}\) be an \(\binom{N}{M}\) tensor.

Then: \(T(S) = S(T) = t_{j_1,\ldots,j_N}^{i_1,\ldots,i_M} s_{i_1,\ldots,i_M}^{j_1,\ldots,j_N}\).

Therefore:

\[ T_p^{(N,M)*} = T_p^{(M,N)} \]
\[ T_p^{(M,N)*} = T_p^{(N,M)} \].
Example

Vectors and one-forms as \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) tensors, respectively.
Tensor Products on the Manifold

Example

Vectors and one-forms as \((\begin{array}{c}1 \\ 0 \end{array})\) and \((\begin{array}{c}0 \\ 1 \end{array})\) tensors, respectively.

Example

Matrices as \((\begin{array}{c}1 \\ 1 \end{array})\) tensors.
Tensor Products on the Manifold

Example

Vectors and one-forms as \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) tensors, respectively.

Example

Matrices as \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) tensors.

Example

Multipartite quantum states as \( \begin{pmatrix} N \\ 0 \end{pmatrix} \) tensors.
Tensor Products on the Manifold

Example

Vectors and one-forms as \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) tensors, respectively.

Example

Matrices as \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) tensors.

Example

Multipartite quantum states as \( \binom{N}{0} \) tensors.

Example

More examples in future lecture (metric tensor, Maxwell stress tensor, etc.).
Tensor Products on the Manifold

Partial Contractions

An \( \binom{N}{M} \) tensor can be converted into an \( \binom{N-K}{M} \) tensor by acting on \( K \) one-forms:

\[
T = t_{j_1,\ldots,j_N}^{i_1,\ldots,i_M} e_{i_1,\ldots,i_M} \rightarrow T' = T(\cdots | \tilde{\tau}_\sigma(1), \cdots, \tilde{\tau}_\sigma(K), \cdots) \\
= t_{j_1,\ldots,j_N}^{i_1,\ldots,i_M} \omega_{i_\sigma(1)} \cdots \omega_{i_\sigma(K)} e_{j_\sigma(K+1),\ldots,j_\sigma(N)}. 
\]
Tensor Products on the Manifold

Partial Contractions

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$$= t_{j_1, \ldots, j_N}^{i_1, \ldots, i_M} \omega_{i_\sigma(1)} \cdots \omega_{i_\sigma(K)} e_{i_1, \ldots, i_M}^{j_\sigma(K+1), \ldots, j_\sigma(N)}.$$

An $\binom{N}{M}$ tensor can be converted into an $\binom{N}{M-K}$ tensor by acting on $K$ tangent vectors:

$$T = t_{j_1, \ldots, j_N}^{i_1, \ldots, i_M} e_{i_1, \ldots, i_M} \rightarrow T' = T(v^{\sigma(1)}, \ldots, v^{\sigma(K)}, \cdots | \cdots)$$

$$= t_{j_1, \ldots, j_N}^{i_1, \ldots, i_M} v_{i_\sigma(1)} \cdots v_{i_\sigma(K)} e_{i_1, \ldots, i_M}^{j_\sigma(K+1), \ldots, j_\sigma(N)}.$$
Vector and Tensor Fields

Definition

A **vector field** $\vec{V}$ on a manifold $\mathcal{M}$ is mapping $\vec{V} : \mathcal{M} \rightarrow T_p(\mathcal{M})$. That is, for every point $p \in \mathcal{M}$, $\vec{V}$ assigns a vector $\mathbf{v} \in T_p(\mathcal{M})$.

Definition

If $\{x^k\}$ is a local coordinate system for a neighborhood of $p$, then we can express

$$\vec{V}(p) = \mathbf{v}(p) = v^k(p) \frac{\partial}{\partial x^k}.$$ 

The components $v^k(p)$ are thus functions on $\mathcal{M}$.

The vector field $\vec{V}$ is **smooth** if the functions $v^k(p) = v^k(x^1, \ldots, x^n)$ are smooth.
Vector and Tensor Fields

Every curve on $\mathcal{M}$ has a tangent vector at every point along the curve. But is the converse true?

Given a vector field $\vec{V}$, is it possible to start at one point $p$ and find a curve whose tangent vectors are specified by $\vec{V}$?
Vector and Tensor Fields

Every curve on $\mathcal{M}$ has a tangent vector at every point along the curve. But is the converse true?

Given a vector field $\overrightarrow{V}$, is it possible to start at one point $p$ and find a curve whose tangent vectors are specified by $\overrightarrow{V}$?

**Definition**

The question is whether there exists a curve $\gamma(\lambda) = (x^1(\lambda), \cdots, x^n(\lambda))$ such that

$$\frac{dx^k}{d\lambda} = v^k(x^1, \cdots, x^n) \quad \forall k = 1, \cdots, n. \quad (1)$$

If the $v^k$ are smooth functions, then a (unique) infinite family of such curves can always be found. They are called the **integral curves** of the vector field. We denote $\overrightarrow{V} = \frac{d}{d\lambda}$ for vector fields with integral curves.
Vector and Tensor Fields

Example

\[ \frac{d}{d\lambda} = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}. \]
Exponentiation of Vector Field

Definition

Let \( \frac{d}{d\lambda} \) be a vector field, and \( x^k(\lambda) \) coordinate values of the integral curves. For some parameter values \( \lambda_0 \) and \( \lambda_0 + \epsilon \), consider the Taylor series evaluated at \( \epsilon = 0 \):

\[
x^k(\lambda_0 + \epsilon) = x^k(\lambda_0) + \epsilon \frac{d}{d\lambda} x^k(\lambda_0) + \cdot + \frac{1}{2} \epsilon^2 \frac{d^2}{d\lambda^2} x^k(\lambda_0)
\]

\[
= \left( 1 + \epsilon + \frac{1}{2} \epsilon^2 \frac{d^2}{d\lambda^2} \right) x^k \bigg|_{\lambda_0}
\]

\[
:= \exp \left( \epsilon \frac{d}{d\lambda} x^k \right) \bigg|_{\lambda_0}.
\]

The operator \( \exp \left( \epsilon \frac{d}{d\lambda} \right) = e^{\epsilon \frac{d}{d\lambda}} \) is called the exponentiation of the vector field. When acting on the \( x^k \), it moves by an \( \epsilon \) along the integral curves (exactly).
Lie Bracket

For a local coordinate system \( \{x^k\} \), each of the \( \frac{\partial}{\partial x^k} \) define a vector field. They are linearly independent and provide a basis for \( T_p(M) \) for each point \( p \).
Lie Bracket

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Notice that the \( \frac{\partial}{\partial x^k} \) commute: \( \left[ \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right] = 0. \)
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Definition

For an \( n \)-dimensional manifold \( M \), a collection of vector fields \( \{\overrightarrow{V}_k\}_{k=1}^n \) is called a \textit{coordinate basis} if \( \overrightarrow{V}_k = \frac{\partial}{\partial y^k} \) for some set of local coordinates \( \{y^k\} \).
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Definition

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For a coordinate basis, the local coordinates \( y^k \) parametrize the integral curves themselves!
Under what conditions do the $\overline{V}_k = \frac{d}{d\lambda_k}$ form a coordinate basis?

A necessary condition is that $\frac{d}{d\lambda_j} = v_j \frac{\partial}{\partial x_j}$ and $\frac{d}{d\lambda_k} = w_k \frac{\partial}{\partial x_k}$ must commute:

$$\frac{d}{d\lambda_j} \frac{d}{d\lambda_k} - \frac{d}{d\lambda_k} \frac{d}{d\lambda_j} = (v_j \frac{\partial}{\partial w_k \frac{\partial}{\partial x_j}} - w_j \frac{\partial}{\partial v_k \frac{\partial}{\partial x_j}}) \frac{\partial}{\partial x_k}.$$
Under what conditions do the $V_k = \frac{d}{d\lambda_k}$ form a coordinate basis?

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$$\frac{d}{d\lambda_j} \frac{d}{d\lambda_k} - \frac{d}{d\lambda_k} \frac{d}{d\lambda_j} = \left( v^j \frac{\partial w^k}{\partial x^j} - w^j \frac{\partial v^k}{\partial x^j} \right) \frac{\partial}{\partial x^k}.$$
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\]

Definition

The commutator \( \left[ \frac{d}{d \lambda_j}, \frac{d}{d \lambda_k} \right] \) is called the **Lie bracket** of the two vector fields. The Lie bracket is a vector field itself.
Example

Consider polar coordinates \( x = r \cos \theta \), \( y = r \sin \theta \) and the vector fields obtained by the rotation:

\[
\left( \frac{\partial}{\partial r}, \frac{\partial}{\partial \tau} \right) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}.
\]
Lie Bracket

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Consider polar coordinates $x = r \cos \theta$, $y = r \sin \theta$ and the vector fields obtained by the rotation:

$$
\left( \frac{\partial}{\partial r}, \frac{\partial}{\partial \tau} \right) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right).
$$

Picture of the commutator.