1. (40 points.) For two functions

\[ A = A(x, p, t), \]
\[ B = B(x, p, t), \]

the Poisson bracket with respect to the canonical variables \( x \) and \( p \) is defined as

\[ [A, B]_{x,p}^{\text{PB}} = \frac{\partial A}{\partial x} \cdot \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \cdot \frac{\partial B}{\partial x}. \]  

Show that the Poisson bracket satisfies the conditions for a Lie algebra. That is, show that

(a) Antisymmetry:

\[ [A, B]_{x,p}^{\text{PB}} = -[B, A]_{x,p}^{\text{PB}}. \]

(b) Bilinearity: (\( a \) and \( b \) are numbers.)

\[ [aA + bB, C]_{x,p}^{\text{PB}} = a[A, C]_{x,p}^{\text{PB}} + b[B, C]_{x,p}^{\text{PB}}. \]

Further show that

\[ [AB, C]_{x,p}^{\text{PB}} = A[B, C]_{x,p}^{\text{PB}} + [A, C]_{x,p}^{\text{PB}} B. \]

(c) Jacobi’s identity:

\[ [A, [B, C]_{x,p}^{\text{PB}}]_{x,p}^{\text{PB}} + [B, [C, A]_{x,p}^{\text{PB}}]_{x,p}^{\text{PB}} + [C, [A, B]_{x,p}^{\text{PB}}]_{x,p}^{\text{PB}} = 0. \]

2. (40 points.) Show that the commutator of two matrices,

\[ [A, B] \equiv AB - BA, \]

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

(a) Antisymmetry:

\[ [A, B] = -[B, A]. \]
(b) Bilinearity: \((a\text{ and } b\text{ are numbers.})\)
\[
\] (10)

Further show that
\[
\] (11)

(c) Jacobi’s identity:
\[
\] (12)

3. (40 points.) Show that the vector product of two vectors, in this problem denoted using
\[
[A, B]_v \equiv A \times B,
\] (13)

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

(a) Antisymmetry:
\[
[A, B]_v = -[B, A]_v.
\] (14)

(b) Bilinearity: \((a\text{ and } b\text{ are numbers.})\)
\[
[aA + bB, C]_v = a[A, C]_v + b[B, C]_v.
\] (15)

Further show that
\[
[A \times B, C]_v = A \times [B, C]_v + [A, C]_v \times B.
\] (16)

(c) Jacobi’s identity:
\[
[A, [B, C]_v] + [B, [C, A]_v] + [C, [A, B]_v] = 0.
\] (17)

4. (50 points.) (Refer Dirac, Sec. 21.)
The product rule for Poisson braket can be stated in the following different forms:
\[
[A_1 A_2, B]_{x,p}^{\text{P.B.}} = A_1 \left[A_2, B\right]_{x,p}^{\text{P.B.}} + \left[A_1, B\right]_{x,p}^{\text{P.B.}} A_2,
\] (18a)
\[
[A, B_1 B_2]_{x,p}^{\text{P.B.}} = B_1 \left[A, B_2\right]_{x,p}^{\text{P.B.}} + \left[A, B_1\right]_{x,p}^{\text{P.B.}} B_2.
\] (18b)

Thus, evaluate, in two different ways,
\[
[A_1 A_2, B_1 B_2]_{x,p}^{\text{P.B.}} = A_1 B_1 \left[A_2, B_2\right]_{x,p}^{\text{P.B.}} + A_1 \left[A_2, B_1\right]_{x,p}^{\text{P.B.}} B_2 + B_1 \left[A_1, B_2\right]_{x,p}^{\text{P.B.}} A_2 + \left[A_1, B_1\right]_{x,p}^{\text{P.B.}} B_2 A_2,
\] (19a)
\[
[A_1 A_2, B_1 B_2]_{x,p}^{\text{P.B.}} = B_1 A_1 \left[A_2, B_2\right]_{x,p}^{\text{P.B.}} + B_1 \left[A_1, B_2\right]_{x,p}^{\text{P.B.}} A_2 + A_1 \left[A_2, B_1\right]_{x,p}^{\text{P.B.}} B_2 + \left[A_1, B_1\right]_{x,p}^{\text{P.B.}} A_2 B_2.
\] (19b)

Subtracting these results, obtain
\[
(A_1 B_1 - B_1 A_1) \left[A_2, B_2\right]_{x,p}^{\text{P.B.}} = [A_1, B_1]_{x,p}^{\text{P.B.}} (A_2 B_2 - B_2 A_2),
\] (20)
Using the definition of the commutation relations,
\[
[A, B] \equiv AB - BA,
\]
thus obtain the relation
\[
[A_1, B_1][A_2, B_2]_{\text{P.B.}} = [A_1, B_1]_{\text{P.B.}}[A_2, B_2].
\]
Since this condition holds for the operators \( A_1 \) and \( B_1 \), independent of the operators \( A_2 \) and \( B_2 \), conclude that
\[
[A_1, B_1] = i\hbar [A_1, B_1]_{\text{P.B.}},
\]
\[
[A_2, B_2] = i\hbar [A_2, B_2]_{\text{P.B.}},
\]
where \( \hbar \) has to be a constant, independent of \( A_1, A_2, B_1, \) and \( B_2 \). This is the connection between the commutator bracket in quantum mechanics and the Poisson bracket in classical mechanics. The imaginary number \( i = \sqrt{-1} \) is necessary because the construction
\[
C = \frac{1}{i}(AB - BA)
\]
is, by construction, Hermitian.

5. (20 points.) Given \( F \) and \( G \) are constants of motion, that is,
\[
[F, H]_{\text{P.B.}} = 0 \quad \text{and} \quad [G, H]_{\text{P.B.}} = 0.
\]
Then, using Jacobi’s identity, show that \([F, G]_{\text{P.B.}}\) is also a constant of motion. Thus, conclude the following:
(a) If \( L_x \) and \( L_y \) are constants of motion, then \( L_z \) is also a constant of motion.
(b) If \( p_x \) and \( L_z \) are constants of motion, then \( p_y \) is also a constant of motion.

6. (30 points.) Hamiltonian for a charge particle in a uniform magnetic field is
\[
H(x, p) = \frac{1}{2m} \left( p - \frac{q}{c} A \right)^2, \quad A = \frac{1}{2} B \times r.
\]
(a) For a constant (homogenous in space) magnetic field \( H \), verify that
\[
A = \frac{1}{2} H \times r
\]
is a possible vector potential. That is, show that
\[
\nabla \times A = B.
\]
(b) Evaluate the Hamilton equations of motion.
(c) Show that

\[ \left[ v^i, v^j \right]_{\text{P.B.}}^{x,p} = \frac{2}{m^2} \mathbf{1} \times \mathbf{B} \equiv \frac{2}{m^2} \varepsilon^{ijm} B_m, \quad (30) \]

which is sometimes expressed in the form

\[ v \times v = \frac{2}{m^2} \mathbf{B} \quad (31) \]

using the fact that the vector product also satisfies the same Lie algebra.