1. (40 points.) Robertson’s generalization of Heisenberg’s uncertainty relation

\[(\delta A)(\delta B) \geq \frac{1}{2}|\langle C \rangle|,\]  

for \((A, B) = (\sigma_x, \sigma_y)\) reads

\[(\delta \sigma_x)(\delta \sigma_y) \geq |\langle \sigma_z \rangle|.\]  

(a) Show that minimal uncertainty states \(|\text{min}\rangle\), characterized by the equality

\[(\delta \sigma_x)(\delta \sigma_y) = |\langle \sigma_z \rangle|,\]  

satisfies matrix equations

\[(\sigma_x \pm i\sigma_y)|\text{min}\rangle = 0.\]  

Hint: Note that \(\sigma_x \sigma_y + \sigma_y \sigma_x = 0\).

(b) Determine the two (normalized) minimum uncertainty states \(|\text{min}\rangle\). Are linear combinations of the two states minimal uncertainty states?

(c) Evaluate \langle \sigma_x \rangle, \langle \sigma_y \rangle, \langle \sigma_z \rangle, \langle \sigma_x^2 \rangle, \langle \sigma_y^2 \rangle, \langle \sigma_z^2 \rangle, \delta \sigma_x, \delta \sigma_y, \text{ and } \delta \sigma_z, \text{ when the system is in the minimum uncertainty state.}\)

(d) Verify Eq. (3).

2. (20 points.) The minimum uncertainty state for Heisenberg’s uncertainty relation

\[\delta q \delta p \geq \frac{1}{2}\]  

in the position eigenbasis is, for \(\langle q \rangle = \langle p \rangle = 0\) and \(\delta q = \delta p = 1/\sqrt{2}\),

\[\psi_0(q') = \frac{1}{\pi^{1/4}} e^{-\frac{1}{2}q'^2}.\]  

Evaluate the minimum uncertainty state in the momentum eigenbasis, \(\psi_0(p')\), by evaluating the integral for the Fourier transform

\[\psi_0(p') = \int_{-\infty}^{\infty} \frac{dq'}{\sqrt{2\pi}} e^{-iq'p'} \psi_0(q').\]
3. (30 points.) Show that

$$\delta(q' - \langle q \rangle) = \lim_{\delta q \to 0} \frac{1}{\sqrt{\pi} \delta q^2} e^{-\left(\frac{q' - \langle q \rangle}{\delta q}\right)^2}$$  \hspace{1cm} (8)$$

is a suitable representation for the Dirac $\delta$-function. That is, verify that it satisfies

$$\delta(q' - \langle q \rangle) \to 0, \quad \text{for} \quad q' \neq \langle q \rangle, \hspace{2cm} (9a)$$

$$\delta(q' - \langle q \rangle) \to \infty, \quad \text{for} \quad q' = \langle q \rangle, \hspace{2cm} (9b)$$

and

$$\int_{-\infty}^{\infty} dq' \delta(q' - \langle q \rangle) = 1. \hspace{1cm} (10)$$